#1.

(10) (a) $\begin{cases} x(\theta) = \theta\cos\theta \\ y(\theta) = \theta\cdot\sin\theta \end{cases} \Longrightarrow \begin{cases} \frac{dx}{d\theta} = \cos\theta - \theta\sin\theta \\ \frac{dy}{d\theta} = \sin\theta + \theta\cos\theta \end{cases}$

$$\Longrightarrow L = \int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_0^{\pi/2}\sqrt{(\cos\theta-\theta\sin\theta)^2+(\sin\theta+\theta\cos\theta)^2}\,d\theta$$

$$= \int_0^{\pi/2}\sqrt{(\cos^2\theta+\sin^2\theta)+\theta^2(\cos^2\theta+\sin^2\theta)}\,d\theta = \int_0^{\pi/2}\sqrt{1+\theta^2}\,d\theta$$

(Let $\theta = \tan t \Rightarrow d\theta = \sec^2 t\,dt$) $= \int \sqrt{1+\tan^2 t}\cdot\sec^2 t\,dt$

$$= \int \sec^3 t\,dt = \left[\frac{1}{2}\sec t\tan t + \frac{1}{2}\ln|\sec t+\tan t|\right] = \frac{1}{2}\left[\left(\sqrt{\theta^2+1}\right)\cdot\theta + \ln\left|\sqrt{\theta^2+1}+\theta\right|\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left\{\sqrt{\frac{\pi^2}{4}+1}\cdot\frac{\pi}{2} + \ln\left|\sqrt{\frac{\pi^2}{4}+1}+\frac{\pi}{2}\right|\right\} - 0\right] = \frac{1}{2}\left(\frac{\pi}{4}\sqrt{\pi^2+4} + \ln\frac{\sqrt{\pi^2+4}+\pi}{2}\right)$$

(10) (b) $\begin{cases} x(t) = 3t - t^3 \\ y(t) = 3t^2 \end{cases} \Longrightarrow \begin{cases} \frac{dx}{dt} = 3-3t^2 \\ \frac{dy}{dt} = 6t \end{cases}$

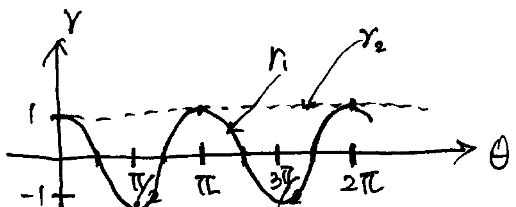
$$\Longrightarrow S = \int 2\pi y(t)\,\underset{\substack{\uparrow \\ \text{arc length}}}{ds} = \int_0^1 2\pi\cdot(3t^2)\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 6\pi\int_0^1 t^2\cdot\sqrt{(3-3t^2)^2+(6t)^2}\,dt$$

$$= 6\pi\int_0^1 t^2\cdot\sqrt{9-18t^2+9t^4+36t^2}\,dt = 6\pi\int_0^1 t^2\cdot\sqrt{(3+3t^2)^2}\,dt = 6\pi\int_0^1(3t^2+3t^4)\,dt$$

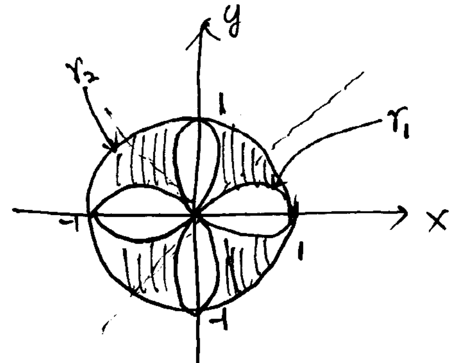
$$= 18\pi\left[\frac{1}{3}t^3+\frac{1}{5}t^5\right]_0^1 = 18\pi\cdot\left(\frac{1}{3}+\frac{1}{5}\right) = \frac{144\pi}{15}$$

#2.

(8) (a)



$\Longrightarrow$



(8)(b) Solve $\cos 2\theta = 1 \implies 2\theta = 0, 2\pi, 4\pi, \ldots$

$\implies \therefore \theta = 0, \pi, 2\pi, \ldots$

(c) $A = 8\int_0^{\pi/4} \frac{1}{2}(r_2^2 - r_1^2)\,d\theta = 4\int_0^{\pi/4}(1-\cos^2 2\theta)\,d\theta = 4\int_0^{\pi/4}\left(1-\frac{1+\cos 4\theta}{2}\right)d\theta$

$= 4\int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\,d\theta = 2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$.

#3

(8) (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\ln n} \longrightarrow$ diverges because

(Consider a conti function $f(x) = \frac{x}{\ln x}$ and apply $\lim_{x\to\infty} \frac{x}{\ln x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x \to \infty$)

(8) (b) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n-1}) = \lim_{n\to\infty} \frac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{(\sqrt{n+1}+\sqrt{n-1})}$

$= \lim_{n\to\infty} \frac{(\cancel{n}+1)-(\cancel{n}-1)}{(\sqrt{n+1}+\sqrt{n-1})} = \lim_{n\to\infty} \frac{2}{\sqrt{n+1}+\sqrt{n-1}} \longrightarrow 0$ $\therefore$ converges to 0

#4. (a) (10) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \overset{\text{by the integral test.}}{=}$ converges!

(Consider a conti, positive, and decreasing function $f(x)$ on $[2,\infty)$

$f(x) = \frac{1}{x(\ln x)^2} \implies f'(x) = \frac{-(\ln x)^2 - 2\ln x}{x^2(\ln x)^4} = \frac{-\ln x(\ln x+2)}{x^2(\ln x)^4}$

$-\ln x(\ln x+2) < 0 \implies \ln x(\ln x+2) > 0 \implies \ln x < -2$ or $\ln x > 0$

$\implies x < \frac{1}{e^2}$ or $x > 1$

$\therefore$ When $x > 1$, $f(x)$ is decreasing.)

Then, for the integral test, consider $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \underset{\substack{u=\ln x \\ du = \frac{1}{x}dx}}{=} \int \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]$

$= -\left[\frac{1}{\ln x}\right]_2^{\infty}$

converges.

(10)
(b) $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ converges by the comparison test.

(Note that $\frac{5+2n}{(1+n^2)^2} < \frac{5+2n}{n^4}$ for $\forall n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{5+2n}{n^4} = \sum_{n=1}^{\infty} \frac{5}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^3}$ converges by p-series test,

thus $\sum_{n=1}^{\infty} \frac{5+2n}{n^4}$ converges.) Then,

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ diverges.

(10)

Consider $\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} > \lim_{n\to\infty} n \longrightarrow \infty \neq 0$

Thus, by the alternating series test,

#5

(a) (10) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = \frac{8}{5} + \frac{8}{5}\cdot\frac{4}{5} + \frac{8}{5}\cdot\frac{4}{5}\cdot\frac{4}{5} + \cdots \longrightarrow \frac{8/5}{1-4/5} = \frac{8/5}{1/5} = 8$

$\therefore a = \frac{8}{5}, \quad r = \frac{4}{5}$

(10) (b) $\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)} = \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right) + \cdots$

(Note: $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$)

$\therefore S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) \longrightarrow \frac{5}{6}$.