Answer key of Test 2 (M2402) F'11

#1.

(a) multiply $(\sqrt{x+1}-\sqrt{x})$ top & bottom.

$$\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx = \int \frac{(\sqrt{x+1}-\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})}\,dx = \int(\sqrt{x+1}-\sqrt{x})\,dx$$

$$= \int\sqrt{x+1}\,dx - \int\sqrt{x}\,dx = \frac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

(b)
$$\int_1^2 \frac{\ln x}{x^2}\,dx = \int \frac{u}{e^u}\cdot du = \int u\cdot e^{-u}\,du \overset{\text{IBP}}{=} -ue^{-u} + \int e^{-u}\cdot 1\,du$$

Let $u=\ln x \Leftrightarrow x=e^u$

$du = \frac{1}{x}dx$

$$= \left[-ue^{-u} - e^{-u}\right] = -\left[(\ln x)\cdot e^{-\ln x} + e^{-\ln x}\right]$$

$$= -\left[\frac{1}{x}\{(\ln x)+1\}\right]_1^2 = -\left\{\left(\frac{1}{2}\ln 2 + \frac{1}{2}\right) - (0+1)\right\}$$

Note: $e^{-\ln x} = \frac{1}{x}$

$$= -\frac{1}{2}\ln 2 + \frac{1}{2}.$$

(c)
$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{1}{(u-1)}\,du = \int \frac{1}{u\cdot(u-1)}\,du$$

Let $u = e^x+1 \Leftrightarrow e^x = u-1$

$du = e^x dx$

$\Rightarrow \frac{1}{(u-1)}du = dx$

$$\Longrightarrow \frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{(A+B)u - A}{u(u-1)}$$

$\therefore A+B=0$

$A=-1$

$\therefore B=1$

$$= \int -\frac{1}{u} + \frac{1}{(u-1)}\,du$$

$$\Downarrow$$

$$= -\ln|u| + \ln|u-1| + C = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{e^x}{e^x+1}\right| + C$$

#2.

$$|E_{S_n}| \le \frac{K(b-a)^5}{180n^4} \quad \text{where } |f^{(4)}(x)| \le K \text{ for } x\in(0,1)$$

$$y' = -3e^{-3x} \Rightarrow y'' = 9e^{-3x} \Rightarrow y''' = -27e^{-3x} \Rightarrow y^{(4)} = 81e^{-3x}$$

$$\Longleftarrow |f^{(4)}(x)| \le 81 \text{ because } f^{(4)}(x) \text{ is decreasing in } (0,1)$$

$$\therefore \frac{81\cdot(1-0)^5}{180n^4} < 0.01$$

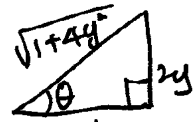
$$\therefore n^4 > \frac{8100}{180} = 45 \qquad \therefore n \geq 3.$$

#3. (a) $x=y^2 \Longrightarrow \frac{dx}{dy}=2y$

$$\therefore L=\int_1^2 \sqrt{1+(2y)^2}\,dy \underset{\uparrow}{=} \int_1^2 \sqrt{1+\tan^2\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta=\frac{1}{2}\int_1^2 \sec^3\theta\,d\theta$$

Let $2y=\tan\theta$

$dy=\frac{1}{2}\sec^2\theta\,d\theta$



$$\left(\text{Note: } \underbrace{\int \sec^3\theta\,d\theta}_{\text{I}}=\int \sec\theta(\sec^2\theta\,d\theta)\overset{\text{IBP}}{=}\sec\theta\tan\theta-\int\tan\theta\cdot(\sec\theta\tan\theta)\,d\theta\right.$$

$$=\sec\theta\tan\theta-\int\tan^2\theta\cdot\sec\theta\,d\theta$$

$$=\sec\theta\tan\theta-\int(\sec^2\theta-1)\cdot\sec\theta\,d\theta$$

$$=\sec\theta\tan\theta-\underbrace{\int\sec^3\theta\,d\theta}_{\text{I}}+\int\sec\theta\,d\theta$$

$$\left.\therefore \text{I}=\frac{1}{2}\Big[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big]+C\right)$$

$$=\frac{1}{4}\Big[\left(\sqrt{1+4y^2}\right)(2y)+\ln\left|\sqrt{1+4y^2}+2y\right|\Big]_1^2=\frac{1}{4}\Big[\{4\sqrt{17}+\ln(\sqrt{17}+4)\}-\{2\sqrt{5}+\ln(\sqrt{5}+2)\}\Big]$$

(b) $$S=\int_1^5 2\pi y\sqrt{1+(y')^2}\,dx=2\pi\int\sqrt{1+4x}\cdot\sqrt{1+\left(\frac{2}{\sqrt{1+4x}}\right)^2}\,dx$$

$y'=\frac{\cancel{4}^{\,2}}{\cancel{2}\sqrt{1+4x}}$

$$=2\pi\int\cancel{\sqrt{1+4x}}\cdot\frac{\sqrt{1+4x+4}}{\cancel{\sqrt{1+4x}}}\,dx=2\pi\int_1^5\sqrt{5+4x}\,dx$$

$$=2\pi\left[\frac{1}{6}(5+4x)^{3/2}\right]_1^5=\frac{\pi}{3}\left[(5+4x)^{3/2}\right]_1^5=4\pi\big[\{125\}-\{27\}\big]=392\pi$$

#4. (a) $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \int_0^1\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx = \int_0^1 \sqrt{x}\,dx - \lim_{T\to 0^+}\int_T^1 \frac{1}{\sqrt{x}}\,dx$

$\displaystyle = \frac{2}{3}x^{\frac{3}{2}}\Big]_0^1 - \lim_{T\to 0^+}\left[2x^{\frac{1}{2}}\right]_T^1 = \frac{2}{3} - 2\cdot\lim_{T\to 0^+}[1-\underbrace{\sqrt{T}}_{\to 0}] = \frac{2}{3}-2 = -\frac{4}{3}$ ∴ converges!

(b) $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{T\to\infty}\int_1^T \frac{1}{(2x+1)^3}\,dx = \lim_{T\to\infty}\left[-\frac{1}{4}(2x+1)^{-2}\right]_1^T = -\frac{1}{4}\lim_{T\to\infty}\left[\underbrace{\frac{1}{(2T+1)^2}}_{\to 0} - \frac{1}{9}\right]$

$\displaystyle = \frac{1}{36}$ ∴ converges.

(c) $\displaystyle\int_1^\infty \frac{1}{x^2+e^{-x}}\,dx < \boxed{\int_1^\infty \frac{1}{x^2}\,dx = \lim_{T\to\infty}\left[-\frac{1}{x}\right]_1^T = -\lim_{T\to\infty}\left[\underbrace{\frac{1}{T}}_{\to 0}-1\right] = 1 \;\therefore \text{converges}}$

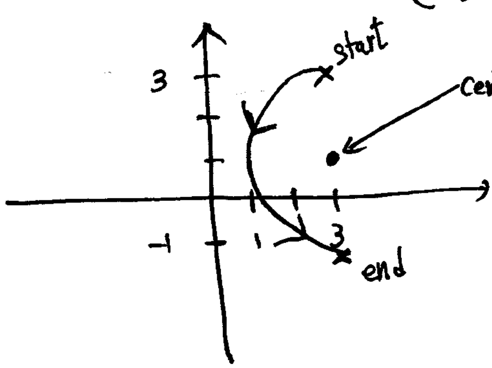
(Note: $e^{-x} > 0$ for $x \in [1,\infty)$)

By the comparison test, $\displaystyle\int_1^\infty \frac{1}{x^2+e^{-x}}\,dx$ also converges.

#5. (a) Rewrite: $\cos t = \dfrac{x-3}{2}$, $\sin t = \dfrac{y-1}{2}$

$\displaystyle\therefore \cos^2 t + \sin^2 t = \left(\frac{x-3}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1. \iff \underbrace{(x-3)^2+(y-1)^2 = 2^2}$

; a circle centered at (3, 1) and the radius = 2.



Note: when $t=\frac{\pi}{2} \Rightarrow \cos\frac{\pi}{2} = 0 = \frac{x-3}{2}$

$\therefore x = 3$

$\sin\frac{\pi}{2} = 1 = \frac{y-1}{2}$

$\therefore y = 3$

When $t = \frac{3\pi}{2} \Rightarrow \cos\frac{3\pi}{2} = 0 = \frac{x-3}{2}$ $\therefore x = 3$

$\sin\frac{3\pi}{2} = -1 = \frac{y-1}{2} \Rightarrow \therefore y = -1$

(b) $\frac{dx}{dt} = -e^{-t} + e^{t}$, $\frac{dy}{dt} = -2$

$$\therefore L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^3 \sqrt{(-e^{-t}+e^{t})^2 + (-2)^2}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\, dt = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt =$$

$$= \int_0^3 \sqrt{(e^{t}+e^{-t})^2}\, dt = \int_0^3 (e^{t}+e^{-t})\, dt = \left[e^{t} - e^{-t}\right]_0^3 = \left\{e^3 - \frac{1}{e^3}\right\} - \{1-1\}$$

$$= e^3 - \frac{1}{e^3}$$