# Answer Key of Test I (M2402) F'11

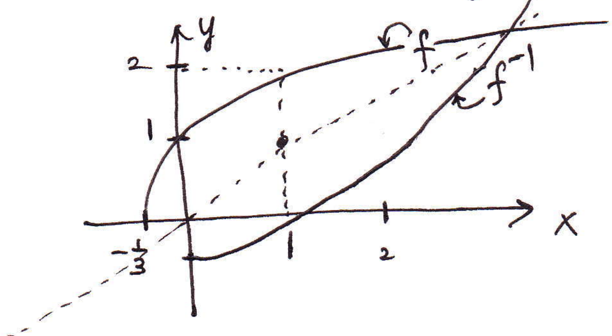
#1. $f(x) = \sqrt{3x+1}$ $\quad$ $\text{domain}_f = [-\frac{1}{3}, \infty)$ , $\text{range}_f = [0, \infty)$

(4 each)

(a) $f: 1-1$ ✓

$y = \sqrt{3x+1} \Longrightarrow y^2 = 3x+1 \Longrightarrow x = \dfrac{y^2-1}{3} \Longrightarrow \therefore f^{-1}(x) = \dfrac{x^2-1}{3}$

(b) $\text{domain}_{f^{-1}} = [0, \infty)$ , $\text{range}_{f^{-1}} = [-\frac{1}{3}, \infty)$




(c) $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{3/2} = 2/3$ .

$f^{-1}(1) = x \Leftrightarrow f(x) = 1 \Longrightarrow \sqrt{3x+1} = 1 \Longrightarrow 3x+1 = 1 \Longrightarrow \therefore x = 0$ .

$f'(x) = \dfrac{3}{2\sqrt{3x+1}} \Longrightarrow f'(0) = 3/2$

#2. $f(x) = x \cdot \ln x$

(4 each)

(a) $f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 > 0 \Longrightarrow \ln x > -1 \Longleftrightarrow x > e^{-1}$ .

(b) $\displaystyle\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{\underset{\uparrow\, LH}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) \longrightarrow 0$

$\displaystyle\lim_{x\to +\infty} x \cdot \ln x = \infty$ .

(c) $f''(x) = \frac{1}{x} > 0 \Longrightarrow x > 0$ .

(d) Note that there exist a local minimum value $-\frac{1}{e}$ when $x = \frac{1}{e}$ .

From (b), we know that $f(\frac{1}{e}) = -\frac{1}{e}$ is the absolute minimum value .

(e) $f'(e) = \ln e + 1 = 2 \implies \therefore$ The eq. of tangent of $f$ at $(e, e)$ is

$$y - e = 2(x - e).$$

#3. (7 each)

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{1+\ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln x}}{\frac{1}{\ln x} + \frac{\ln x}{\ln x}} = \lim_{x\to\infty} \frac{1}{\frac{1}{\ln x}+1} \longrightarrow 1.$

(b) $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}} \Rightarrow$

Let $y = x^{\frac{1}{x}}$. Take $\ln$ both $\implies \ln y = \ln x^{\frac{1}{x}} = \frac{1}{x}\ln x$

$\underset{\text{differentiate both w.r.t. } x}{\implies} \frac{1}{y}\cdot y' = \left(-\frac{1}{x^2}\right)\ln x + \frac{1}{x}\left(\frac{1}{x}\right) = \frac{1}{x^2}(1-\ln x)$

$\implies \therefore y' = y\cdot\left[\frac{1}{x^2}(1-\ln x)\right] = x^{\frac{1}{x}}\cdot\frac{1}{x^2}\cdot(1-\ln x).$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{e^x} = \lim_{x\to\infty} \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x} - \frac{1}{e^{2x}}}{\frac{2e^x}{e^x}}$

$\displaystyle = \lim_{x\to\infty} \frac{1-\frac{1}{e^{2x}}}{2} \longrightarrow \frac{1}{2}.$

#4. (8 each)

(a) $y = 3^{-2x} \implies \ln y = \ln 3^{-2x} = -2x\cdot\ln 3 \implies \frac{1}{y}\cdot y' = -2\ln 3$

$\implies \therefore y' = y\cdot(-2\ln 3) = -2\cdot 3^{-2x}\cdot\ln 3$

(b) $y = x^{\frac{1}{x}} \implies \ln y = \frac{1}{x}\ln x \implies \frac{1}{y}\cdot y' = \left(-\frac{1}{x^2}\right)\ln x + \frac{1}{x}\left(\frac{1}{x}\right)$

$= \frac{1}{x^2}\cdot(1-\ln x)$

$\implies \therefore y' = y\cdot\frac{1}{x^2}\cdot(1-\ln x)$

$= x^{\frac{1}{x}}\cdot\frac{1}{x^2}\cdot(1-\ln x) = x^{\frac{1}{x}-2}\cdot(1-\ln x)$

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#5. (8 each)

(a) $\int x\cdot\sqrt{25+x^2}\,dx \underset{\uparrow}{=} \int \sqrt{u}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{\frac{3}{2}}\right)+C$

$u=25+x^2$
$du=2x\,dx$
$\frac{1}{2}du=x\,dx$

$= \frac{1}{3}(25+x^2)^{3/2}+C$.

(b) $\int_0^1 x\cdot e^{-x^2}dx \underset{\uparrow}{=} \int e^u\cdot\left(-\tfrac{1}{2}du\right) = -\frac{1}{2}\int e^u du = \left[-\frac{1}{2}e^u\right] = -\frac{1}{2}\left[e^{-x^2}\right]_0^1$

$u=-x^2$
$du=-2x\,dx$
$-\frac{1}{2}du=x\,dx$

$= -\frac{1}{2}\left[\frac{1}{e}-1\right] = \frac{1}{2}\left[1-\frac{1}{e}\right]$.

(c) $\int \frac{(\ln x)^2}{x}dx \underset{\uparrow}{=} \int u^2du = \frac{1}{3}u^3+C = \frac{1}{3}(\ln x)^3+C$

$u=\ln x$
$du=\frac{1}{x}dx$

(d) $\int \frac{t}{\sqrt{1-t^4}}dt \underset{\uparrow}{=} \int\frac{\frac{1}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u+C$

$u=t^2$
$du=2t\,dt$
$\frac{1}{2}du=t\,dt$

$= \frac{1}{2}\sin^{-1}(t^2)+C$.

(e) $\int_0^2\frac{1}{\sqrt{4+x^2}}dx = \int_0^2\frac{1}{2\sqrt{1+\left(\frac{x}{2}\right)^2}}dx \underset{\uparrow}{=} \frac{1}{\cancel{2}}\int\frac{\cancel{2}du}{\sqrt{1+u^2}} = \int\frac{1}{\sqrt{1+u^2}}du = \left[\sinh^{-1}(u)\right]$

$u=\frac{x}{2}$
$du=\frac{1}{2}dx$
$2du=dx$

$= \left[\sinh^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \sinh^{-1}(1)-\sinh^{-1}(0) = \ln(1+\sqrt{2})-0 = \ln(1+\sqrt{2})$ ///

$\sinh^{-1}(0)=B \iff 0=\frac{e^B-e^{-B}}{2} \Rightarrow e^B=e^{-B} \Rightarrow \therefore B=0$

$\sinh^{-1}(1)=A \iff 1=\sinh A=\frac{e^A-e^{-A}}{2} \Rightarrow e^A-e^{-A}=2$

$\Rightarrow (e^A)^2-2e^A-1=0$

$\Rightarrow \therefore e^A=\frac{2\pm\sqrt{4-4(1)(-1)}}{2}=\frac{2\pm2\sqrt{2}}{2}$

$\therefore e^A=1+\sqrt{2} \iff A=\ln(1+\sqrt{2})$

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#6 $\tanh(\ln x) = \dfrac{x^2-1}{x^2+1}$

(8)

$$LHS = \frac{\frac{e^{\ln x}-e^{-\ln x}}{\cancel{2}}}{\frac{e^{\ln x}+e^{-\ln x}}{\cancel{2}}} = \frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}} = \frac{x-\frac{1}{x}}{x+\frac{1}{x}} = \frac{x^2-1}{x^2+1} = RHS$$