The Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches** a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Now let's put this definition to work.

Prove that $\lim_{x \to 3}(7x + 2) = 23$.

(First we must guess a value for δ.)
Let ϵ be any positive number. We want to find a number δ such that
$|(7x + 2) - 23| < \epsilon$ whenever $0 < |x - 3| < \delta$.
Now, $|(7x + 2) - 23| = |7x - 21| = |7(x - 3)| = |7| \cdot |x - 3| = 7|x - 3|$.
So, we want $7|x - 3| < \epsilon$ whenever $0 < |x - 3| < \delta$
that is $|x - 3| < \epsilon/7$ whenever $0 < |x - 3| < \delta$.
Choose $\delta = \epsilon/7$.
(Now, we are ready to write a proof that is to show this number δ works.)
$Proof$: Given $\epsilon > 0$, choose $\delta = \epsilon/7$. Then, whenever $0 < |x - 3| < \delta$,
$|(7x + 2) - 23| = |7x - 21| = |7(x - 3)| = |7| \cdot |x - 3| =$
$7|x - 3| < 7\delta = 7 \cdot \epsilon/7 = \epsilon$.
Thus, $|(7x + 2) - 23| < \epsilon$ whenever $0 < |x - 3| < \delta$.
Therefore, $\lim_{x \to 3}(7x + 2) = 23$.

Go to the next page and you practice.

Prove that $\lim_{x \to 2}(5x+8) = 18$.

(First we must guess a value for δ.)

Let ϵ be any positive number. We want to find a number δ such that

____________________ whenever ____________________.

Now, ____.

So, we want ____________________ whenever ____________________

that is ____________________ whenever ____________________.

Choose $\delta =$ _______.

(Now, we are ready to write a proof that is to show this number δ works.)

$Proof$: Given $\epsilon > 0$, choose $\delta =$ _______. Then, whenever __________________,

____

____.

Thus, _________________ whenever _________________.

Therefore, _________________.

Let's do one more.

Prove that $\lim_{x \to -1}(3x-7) = -10$.