Some Elementary Lower Bounds on the Matching Number of Bipartite Graphs

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Abstract

In this paper we present some elementary results on the matching number of bipartite graphs. Most of the results are lower bounds on the matching number of bipartite graphs in terms of various degrees of the graph and others involve various subsets of vertices of the graph. For example, we prove that for any subset S of vertices of a bipartite graph, the matching number is at least the minimum of the smallest degree of a vertex appearing in S and the cardinality of S.

1 Introduction

In this paper all graphs are assumed to be simple and finite. The notation G, $V(G)$ and $E(G)$ is used to denote a graph, its vertex set and edge set. A graph G is a **bipartite graph** if its vertices can be partitioned into two disjoint sets X and Y such that $X \cup Y = V(G)$ and every edge of the graph is of the form (x, y) with $x \in X$ and $y \in Y$. An **X, Y-bigraph** is a bipartite graph with partition X and Y. A subset of the edges of G is called a **matching** if no two edges have a common endpoint. The size of a largest matching is the **matching number** of a graph G, which we denote by $\mu(G)$.

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Most of the lower bounds on the matching number of bipartite graphs presented in this paper were conjectured by the first author's conjecture-making program, Graffiti.pc (see [3] for program description). This program was inspired by Siemion Fajtlowicz's conjecture-making program, Graffiti (see [5] for description). Since a majority of the results involve various degree invariants of the graph, we devote a short section to those definitions and notation. All other necessary definitions will be presented before referenced.

2 Definitions on Degrees of a Graph

The **degree of a vertex** v of G, denoted by $deg(v)$, is the number of vertices adjacent to v. By the **ordered degree sequence** of a graph we mean the degree sequence listed in non-decreasing order. We denote the **maximum degree of** G by $\Delta(G)$, the **minimum degree of** G by $\delta(G)$, the **second largest degree of the degree sequence of** G by $\Sigma(G)$, and the **second smallest degree of the degree sequence of** G by $\sigma(G)$; the latter is the second entry of the ordered degree sequence. Note that the ordered degree sequence of G can be listed as $\delta(G), \sigma(G), ..., \Sigma(G), \Delta(G)$.

The **median of the degree sequence** is the $(k+1)^{th}$ entry of the ordered degree sequence in the case that the ordered degree sequence has $2k+1$ terms; otherwise, in case G has $2k$ vertices, it is the average of the $k^{th}$and $(k+1)^{th}$ entries of the ordered degree sequence. The **number of distinct degrees of** G, denoted $dd(G)$, is the number of distinct integers occurring in the degree sequence. Let S be a subset of the vertices of G. The **neighborhood of** S, denoted $N(S)$, is the set of vertices that are adjacent to some vertex of S. Lastly, we put $\Delta(S) = \text{maximum}\{deg(v)|v \in S\}$ and $\delta(S) = \text{minimum}\{deg(v)|v \in S\}$.

3 Applications of Berge's Theorem

The organization of this paper was partly motivated by the classic results that seemed most useful in proving particular conjectures. As such, we first state the classic result of C. Berge and the needed terminology.

Let M be a matching of G. An M**-alternating path** is a path whose edges alternate between edges in M and edges not in M. A vertex v of G is **saturated by** M if it is the endpoint of an edge of M; otherwise, vertex v is **unsaturated by** M. An M**-augmenting path** is an M-alternating path that begins and ends with M-unsaturated vertices.

Theorem 1. (Berge [1]) *A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.*

Our first theorem was inspired by three of Graffiti.pc's conjectures (numbered 114, 115 and 116 in [4]); those conjectures now follow from our theorem.

Theorem 2. *Let G be a bipartite graph. Then*

$$\mu(G) \geq maximum\{minimum\{|S|, \delta(S)\} : S \subseteq V(G)\}.$$

Proof. Let $S \subseteq V(G)$; recall that $\delta(S) = minimum\{deg(v) : v \in S\}$. Let M be the set of edges of a maximum matching. We will assume that the set S has no isolated vertices since otherwise $\delta(S) = 0$. If $\mu(G) = 1$, then G is a star and any subset on more than one vertex will contain a vertex of degree one, in which case the result clearly follows. Thus, we can assume $\mu(G) \geq 2$. By way of contradiction suppose that $\mu(G) < |S|$ and $\mu(G) < \delta(S)$ for some $S \subset V(G)$. Since G is bipartite and each vertex of S is assumed to have degree at least $\mu(G) + 1$, each vertex of S has an unsaturated neighbor. From this we see that the vertices in S are saturated by M, otherwise there is an unsaturated vertex of S adjacent to an another unsaturated vertex, contradicting that M is maximum. Since all vertices of S are saturated, the assumption that S has at least $\mu(G) + 1$ vertices implies that there exists an edge (u, v) of M with both endpoints in S. By assumption, the degrees of the endpoints u and v are each at least $\mu(G)+1$. Since G is bipartite, each of the endpoints has a neighbor not saturated by M that yield an M-augmenting path. Hence, by Berge's Theorem we have reached a contradiction to the assumption that M is maximum. □

From Theorem 2 with $S = V(G)$, it follows that the matching number of a bipartite graph is at least its minimum degree; for $S = V(G) - \{v\}$ where v is a vertex of minimum degree, it follows that the matching number of a bipartite graph is at least $\sigma(G)$ (the second smallest degree of the ordered degree sequence); the latter, was conjectured by Graffiti.pc (numbered 118 in [4].) If one continues to construct such a set S until, $|S| = \frac{n+1}{2}$, when $n = |V(G)|$ is odd, then it follows that the matching number is at least the median of the degree sequence. In Theorem 3, we prove that if G is a connected X, Y-bigraph such that $\mu(G) < |X| \leq |Y|$, then $\mu(G)$ is at least one plus the median of the ordered degree sequence.

Lemma 1. *Let G be a connected X, Y bigraph such that $|X| \leq |Y|$. If $\mu(G) < |X|$, then there are at most $\mu(G)$ vertices of degree at least $\mu(G)$.*

Proof. Let M be a maximum matching of G and let $V(M)$ be the set of vertices incident to edges of M. Assume $\mu(G) < |X|$. Let $S = \{v : deg(v) \geq \mu(G)\}$. By way of contradiction, assume $|S| \geq \mu(G) + 1$. Observe that by the assumptions $\mu(G) < |X| \leq |Y|$, it follows that

(*) there exists at least one M-unsaturated vertex in X and at least one M-unsaturated vertex in Y.

Next, we show that every vertex in S is M-saturated. Suppose $x \in S \cap X$ and that x is not saturated. Since the degree of an M-unsaturated vertex is at most $\mu(G)$ and $x \in S$, it follows that $deg(x) = \mu(G)$. This implies that x is adjacent to each vertex of $V(M) \cap Y$. By (*) there is an M-unsaturated vertex y in Y. Since the graph is assumed to be connected, y is adjacent to at least one M-saturated vertex x'. Vertex x' is incident to an edge (x', y') of the matching. This results in an M-augmenting path $x - y' - x' - y$. The case that $x \in S \cap Y$, is argued similarly.

Since the vertices of S are M-saturated and we have assumed that $|S| \geq \mu(G) + 1$, there exists an edge (x, y) of the matching whose endpoints have degree at least $\mu(G)$. Note that x and y cannot both have M-unsaturated neighbors otherwise G has an M-augmenting path.

In the case that one of the endpoints of the edge (x, y) has an M-unsaturated neighbor, first assume that vertex $x \in X$ has unsaturated neighbor y'. By observation (*) we have M-unsaturated vertex x' in X. Since the graph is connected and M is maximum, x' must have at least one saturated neighbor v in Y. Vertex v is incident to an edge (v, u) of the matching. Since y has no M-unsaturated neighbor and the degree of y is at least $\mu(G)$, y is adjacent to each vertex of $V(M) \cap X$. This results in an M-augmenting path $y' - x - y - u - v - x'$. The case that $x \in Y$ is argued similarly.

Lastly, in the case that neither of the endpoints of the edge (x, y) has an M-unsaturated neighbor, note that $N(X) \cup N(Y) = V(M)$. By observation (*) we have M-unsaturated vertices $x' \in X$ and $y' \in Y$. Since G is connected and M is maximum, vertex x' is adjacent to at least one saturated vertex v. Vertex v is incident to an edge (v, u) of the matching. Similarly, y' is adjacent to at least one saturated vertex s. If s is the same as u, then clearly G has an M-augmenting path. So suppose $s \neq u$. Vertex s is incident to an edge (s, p) of the matching. This results in an M-augmenting path $x' - v - u - y - x - p - s - y'$. □

The following two results were conjectured by Graffiti.pc and numbered 124 and 125 in [4].

Theorem 3. *Let G be a connected X, Y-bigraph such that $|X| \leq |Y|$. Then*

$$\mu(G) \geq minimum\{1 + median(G), |X|\}.$$

Proof. Let $|V(G)| = n$ and assume $\mu(G) < |X|$. Then clearly $n \geq 2\mu(G) + 2$. Let us consider the degrees of G in nondecreasing order

$$d_1 \leq ... \leq d_{\mu(G)} \leq d_{\mu(G)+1} \leq ... \leq d_{n-\mu(G)} \leq d_{n-\mu(G)+1} \leq ... \leq d_n.$$

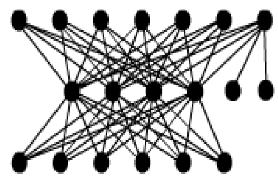

Figure 1: Counterexample to Conjecture 1.

Observe that since $n \geq 2\mu(G)+2$, the median degree is not determined from the $\mu(G)$ vertices with indices greater than $n-\mu(G)$; thus by Lemma 1, the median will be at most $\mu(G)-1$ or equivalently $\mu(G) \geq 1+median(G)$. □

Theorem 4. *Let G be a connected X,Y-bigraph such that $|X| \leq |Y|$. Then*

$$\mu(G) \geq minimum\{1+k, |X|\},$$

where k is the $(n-|X|+1)th$ degree of the degree sequence in nondecreasing order.

Proof. Let $|V(G)| = n$ and assume $\mu(G) < |X|$. Let l be the number of vertices which have degree at least $\mu(G)$, then by Lemma 1, $l \leq \mu(G) \leq |X|-1$. Let us consider the degrees of G in nondecreasing order

$$d_1 \leq ... \leq d_{\mu(G)} \leq ... \leq d_{n-|X|} \leq d_{n-|X|+1} \leq d_{n-|X|+2} \leq ... \leq d_n.$$

Then, the $|X|-1$ vertices whose indices are greater than or equal to $n-|X|+2$ may have degree at least $\mu(G)$, but the one with $n-|X|+1$ will not. Thus, k will be at most $\mu(G)-1$ or equivalently $\mu(G) \geq 1+k$. □

The median of the degree sequence was not the only measure of central tendency to appear in conjectures; however, it turned out that if one will replace the median of the degree sequence with the average of degrees in the statement of Theorem 3, the statement is false.

Conjecture 1. (Graffiti.pc 123 [4]) *Let G be a connected X,Y-bigraph such that $|X| \leq |Y|$. Then*

$$\mu(G) \geq minimum\{\lfloor(1+average\ of\ degrees)\rfloor, |X|\}.$$

The graph in Figure 1 is a counterexample to Conjecture 1; the matching number is five, $|X| = 6$ and the floor of one more than the average of degrees is also 6. This counterexample is a member of the following family of counterexamples to Conjecture 1.

Let $r > k \geq 3$ and H be a complete bipartite graph $K_{kr+1,r}$. Let y be a vertex in the partite set of cardinality $kr+1$ of H. A new graph G is constructed by adding two vertices x_1 and x_2 to H and adding two edges (x_1, y) and (x_2, y) to H. Let M be a maximum matching of G. Since x_1 or x_2 is unsaturated by M, the matching number of G is $r+1$. On the other hand, the average degree of G is

$$\frac{(kr+1)r + 2 + rkr + (r+2)}{(k+1)r+3} = \frac{2kr^2 + 2r + 4}{(k+1)r+3}.$$

Now

$$\lfloor \frac{2kr^2 + 2r + 4}{(k+1)r+3} \rfloor = \lfloor r + \frac{(k-1)r^2 - r + 4}{(k+1)r+3} + 1 \rfloor \geq r + 2.$$

Our last result of this section also originated as a conjecure of Graffiti.pc (number 119 in [4]).

Theorem 5. *Let G be a connected X, Y-bigraph such that $|X| \leq |Y|$. Then*

$$\mu(G) \geq minimum\{2\sigma(G), |X|\}.$$

Proof. Let M be a maximum matching of G. By way of contradiction assume that $\mu(G) = |M| < |X|$ and that $\mu(G) = |M| < 2\sigma(G)$. The assumptions that $\mu(G) < |X| \leq |Y|$ imply that there exists a vertex x in X that is unsaturated by M and a vertex y in Y that is unsaturated by M. Observe that since M is maximum, all neighbors of x and y are among the vertices of M.

One of the vertices x or y is of degree less than $\frac{1}{2}\mu(G)$, for otherwise x and y are incident to a common edge of M, which by Berge's Theorem contradicts that M is maximum. Suppose that $deg(x) < \frac{1}{2}\mu(G)$. In this case, by the assumption that $\mu(G) = |M| < 2\sigma(G)$ we see that every vertex, other than x, is of degree at least $\frac{1}{2}\mu(G)+1$. Since the graph is assumed to be connected, x must be adjacent to a vertex that is incident to an edge of M. Let y' be such a vertex, and let x' be its matched vertex in M. Clearly, x' is in X and y' is in Y. Further, observe that all neighbors of x' are among the vertices incident to edges of M, otherwise path $x - y' - x' - v$ determines an M-augmenting path, where v is a neighbor of x' not in M. Since the degrees of x' and y are at least $\frac{1}{2}\mu(G) + 1$, they are incident to a common edge of M, say (u, v) with u in Y and v in X. In this case, the path $x - y' - x' - u - v - y$ is an M-augmenting path providing a contradiction to Berge's Theorem. The case that y is the vertex of degree less than $\frac{1}{2}\mu(G)$ is argued similarly. □

4 Applications of Hall's Theorem

This short section involves two of Graffiti.pc's conjectures (112 and 113 in [4]) that are direct applications of the classic result known as Hall's Theorem.

A **perfect matching** in a graph is one that saturates all vertices of the graph. A matching M in an X, Y-bigraph is a **complete matching of** X **into** Y if every vertex of X is saturated by M.

Theorem 6. (Hall [6]) *If G is an X, Y-bigraph, then G has a complete matching of X into Y if and only if $|N(S)| \geq |S|$ for all subsets S of X.*

Proposition 1. *Let G be a nonempty bipartite graph on at least two vertices. Let M be the set of vertices of maximum degree of G. Then*

$$\mu(G) \geq \lceil |M|/2 \rceil.$$

Proof. Let G be an X, Y-bigraph with maximum degree $\Delta(G)$. Let M be the set of vertices of maximum degree of G. Let $X^* = M \cap X$ and let $Y^* = M \cap Y$. Clearly, $|M| = |X^*| + |Y^*|$. Let $S \subseteq X^*$. Since the vertices of $N(S)$ are incident with $\Delta(G) \cdot |S|$ edges and at most maximum $\Delta(G) \cdot |N(S)|$ edges, $|N(S)| \geq |S|$. Thus, Hall's Theorem (applied to the subgraph induced by $X^* \cup N(X^*)$) implies that there is a matching that saturates the vertices of X^*. A similar argument yields that there is a matching that saturates the vertices of Y^*. Thus, $\mu(G) \geq |X^*|$, and $\mu(G) \geq |Y^*|$ from which the result follows. □

A standard exercise for applying Hall's theorem is to show that a degree-regular bipartite graph has a perfect matching; this exercise follows from Proposition 1. Graffiti.pc also conjectured that for connected graphs the matching number is at least half of the number of distinct degrees (see 113 in [4]); Craig Larson communicated a proof to the authors. Lastly, we note that for connected bipartite graphs the proof of the latter relation is similar to the application of Hall's Theorem given in the proof of Proposition 1.

5 On the Ratio of Edges to Degree

Let S be a subset of the vertices of G. The **neighborhood of** S, denoted $N(S)$ is the set of vertices that are adjacent to some vertex of S. The set $N(S) - S$ is the set difference of the sets $N(S)$ and S, which is the set of neighbors of vertices of S that are not members of S.

Graffiti.pc rediscovered the exercise that for any bipartite graph, $\mu(G) \geq \frac{|E(G)|}{\Delta(G)}$, which we state here as Proposition 2 and is found for instance in

[8]. Graffiti.pc also conjectured (see number 118 in [4]) that the matching number of a connected bipartite graph is at least the ratio of the number of edges of G with exactly one endpoint that is a minimum degree vertex to the minimum degree.

Proposition 2. ([8], p. 121) *Let G be a bipartite graph. Then*

$$\mu(G) \geq \frac{|E(G)|}{\Delta(G)}.$$

Theorem 7. *Let G be a connected bipartite graph on at least 2 vertices. Let A be the set of vertices of minimum degree in G. Then*

$$\mu(G) \geq \frac{|N(A) - A|}{\delta(G)}.$$

Proof. Let G be a connected X, Y-bigraph on at least two vertices with $\delta = \delta(G)$. Let A be the set of vertices of minimum degree in G. Let $X^* \subseteq A \cap X$ such that $N(X^*) \cap A = \emptyset$ (that is X^* consists of vertices of X that are of minimum degree but whose neighbors are not of minimum degree); similarly, we let $Y^* \subseteq A \cap Y$ such that $N(Y^*) \cap A = \emptyset$.

Let M be a largest matching from X^* to $N(X^*)$. Let $V(M)$ be the vertices of the matching. We next show that

$$N(V(M) \cap X) = N(X^*). \tag{1}$$

Since $(V(M) \cap X) \subseteq X^*$, clearly $N(V(M) \cap X) \subseteq N(X^*)$. On the other hand, suppose that there is a vertex v in $N(X^*)$ that is not in $N(V(M) \cap X)$. This implies that v is adjacent to a vertex u in X^* but not adjacent to any vertex in $(V(M) \cap X)$, which contradicts our assumption that M is maximum.

By (1) we see that

$$\frac{|N(X^*)|}{\delta} = \frac{|N(V(M) \cap X)|}{\delta} \leq \frac{\delta |V(M) \cap X|)}{\delta} = |M|. \tag{2}$$

Let N be a largest matching from Y^* to $N(Y^*)$. One can similarly argue that

$$\frac{|N(Y^*)|}{\delta} \leq |N|. \tag{3}$$

Since X^* and $N(Y^*)$ are disjoint sets, and the sets Y^* and $N(X^*)$ are also disjoint, it is easily seen that $M \cup N$ is a matching of size $|M \cup N| =$

$|M| + |N|$. Thus, by (2) and (3) we see that

$$\begin{aligned} |M \cup N| &\geq \frac{|N(X^*)| + |N(Y^*)|}{\delta} \\ &= \frac{|N(X^*) \cup N(Y^*)|}{\delta} \\ &= \frac{|N(X^* \cup Y^*)|}{\delta}. \end{aligned}$$

Let $A^* = X^* \cup Y^*$, and $M^* = M \cup N$. By construction $N(A^*)$ contains no vertices of minimum degree and thus with this notation and the above we have the following,

$$|M^*| \geq \frac{|N(A^*) - A|}{\delta}. \tag{4}$$

We will extend the set A^* to A by considering the neighborhoods of vertices of $A - (X^* \cup Y^*)$ one at a time and in the process demonstrate that there is a matching whose size is at least $|N(A) - A|/\delta$. Note that the set A is the union of four disjoint sets, Y^*, X^*, $(A \cap X) - X^*$ and $(A \cap Y) - Y^*$.

Let v be a vertex in the set $(A \cap X) - X^*$. If $N(v)$ is contained in $A \cap Y$, then adding the vertex v to set A^* contributes at most δ vertices (of minimum degree) to $N(A^*)$ which results in no net change in the quantity $|N(A^*) - A|$. Similarly, adding to A^* a vertex v of the set $(A \cap Y) - Y^*$ such that $N(v)$ is contained in $A \cap X$ results in no change to the quantity $|N(A^*) - A|$. Thus, we assume that all vertices of minimum degree whose neighborhoods are entirely contained in A are now members of A^* and that the relation in (4) is preserved.

Let v be a vertex of $(A \cap X) - X^*$ whose neighborhood intersects both A and its complement. If the intersection of $N(v)$ with the complement of A is contained in $N(A^*)$ then adding vertex v to A^* results in no net change in $|N(A^*) - A|$ since those neighbors would have already been counted. On the other hand if the intersection of $N(v)$ with the complement of A is not entirely contained in $N(A^*)$, say u is such a neighbor of vertex v, then adding edge (u, v) to M^* increases its size by one, and adding vertex v to A^* results in an increase in $|N(A^*) - A|$ by at most δ, which increases the ratio $|N(A^*) - A|/\delta$ by at most one. □

6 Closing Comments

Once Graffiti.pc was queried for lower bounds on the matching number of connected bipartite graphs, it responded with 19 conjectures. The two conjectures that remain open are listed next.

Conjecture 2. (Graffiti.pc 129 [4])*Let G be a connected X, Y-bigraph such that $|X| \leq |Y|$. Let $\Delta(Y)$ be the maximum degree of Y. If $\Delta(Y) \neq 1$,*

$$\mu(G) \geq \frac{|X|}{\Delta(Y) - 1}.$$

Conjecture 3. (Graffiti.pc 130 [4]) *Let G be a connected X, Y-bigraph on at least three vertices such that $|X| \leq |Y|$. Let Σ be the 2^{nd} largest degree of the degree sequence in nondecreasing order. If $\Sigma \neq 1$,*

$$\mu(G) \geq \frac{|X|}{\Sigma - 1}.$$

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