Bounds on the k-Domination Number of a Graph

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Abstract

The k-domination number of a graph is the cardinality of a smallest set of vertices such that every vertex not in the set is adjacent to at least k vertices of the set. We prove two bounds on the k-domination number of a graph, inspired by two conjectures of the computer program Graffiti.pc. In particular, we show that for any graph with minimum degree at least $2k - 1$, the k-domination number is at most the matching number.

1. Introduction

For a positive integer k, a *k-dominating set* of a graph G is a set S of vertices such that every vertex in $V(G) \setminus S$ has at least k neighbors in S. For a graph G, the minimum cardinality of a k-dominating set is called the *k-domination number* of G, and is denoted $\gamma_k(G)$. This invariant was introduced by Fink and Jacobson [6], and has been studied by a number of authors including [2, 4, 5, 7, 8, 9, 10].

We will use some standard terminology from graph theory, for which we refer the reader to [1]. The *independence number* of a graph G is the cardinality of an independent set of maximum size, and will be denoted $\alpha(G)$. The *matching number* of a graph G is the cardinality of a matching of maximum size in G, and will be denoted $\alpha'(G)$.

If S is a set of vertices of G, then $G[S]$ will denote the subgraph of G induced by S, and $G - S$ will denote the subgraph of G induced by $V(G) \setminus S$. The degree of a vertex v will be denoted $d(v)$, and the minimum degree of G will be denoted $\delta(G)$.

The following result is due to Caro and Roditty [2]:

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Theorem 1. *Let r and k be positive integers. Let G be a graph of order n where $\delta(G) \geq \frac{r+1}{r}k - 1$. Then*

$$\gamma_k(G) \leq \frac{r}{r+1}n.$$

We will need the $r = 1$ version of this theorem. Namely:

Corollary 2. *Let G be a graph of order n where $\delta(G) \geq 2k - 1$. Then*

$$\gamma_k(G) \leq n/2.$$

In this note, we will improve and generalize Corollary 2. Our first result is the following:

Theorem 3. *Let k be a positive integer, and G a graph of order n. Let $H \subseteq V(G)$ be the set of vertices of degree less than $2k - 1$. Then*

$$\gamma_k(G) \leq \alpha'(G - H) + |H|.$$

If we suppose that H is empty, we get the following succinct result:

Corollary 4. *Let k be a positive integer. For any graph G with $\delta(G) \geq 2k - 1$,*

$$\gamma_k(G) \leq \alpha'(G).$$

To see that equality can be achieved in the corollary above, even for graphs that do not have perfect matchings, consider a complete bipartite graph with $2k - 1$ vertices in one part and more than $2k - 1$ vertices in the other part.

Our second result is the following:

Theorem 5. *Let k be a positive integer, and G a graph of order n. Suppose that in G no two vertices of degree less than $2k-2$ are adjacent. Let $H \subseteq V(G)$ be the set of vertices of degree less than $2k - 1$. Then*

$$\gamma_k(G) \leq \frac{n + \alpha(G[H])}{2}.$$

Complete graphs of order $2k - 1$ not only demonstrate that this bound is sharp for every k, but also provide examples where this bound is sharp while Theorem 3 is very weak and Corollary 2 cannot even be applied. Now, if we suppose $k = 2$ and the graph is bipartite, we get the following result of Fujisawa et al. [7]:

Corollary 6. *If G is a bipartite graph, then*

$$\gamma_2(G) \leq \tfrac{3}{2}\alpha(G).$$

These results were inspired by two conjectures of the computer program Graffiti.pc. The program conjectured the special cases of Theorems 3 and 5 where $k = 2$ (see Conjectures 388 and 392a of [3]), and these conjectures were announced at the Southeastern Conference on Combinatorics, Graph Theory and Computing, held in Boca Raton, March 2010.

2. Proof of Theorem 3

We need the following folklore result:

Lemma 7. *For any graph G, $V(G)$ can be partitioned into two parts S and T, such that each vertex v in S has at least $d(v)/2$ neighbors in T, and each vertex $w \in T$ has at least $d(w)/2$ neighbors in S.*

Proof. Consider the partition of $V(G)$ into S and T such that the number of edges between S and T is maximized. Then any vertex must have at least half its neighbors in the other part. □

If G is a graph with $\delta(G) \geq 2k-1$ then by Lemma 7 we can partition $V(G)$ into two parts S and T so that each vertex has at least k neighbors in the other part. Corollary 2 is then an easy consequence: both S and T are k-dominating sets, and at least one of them has size at most $n/2$.

Proof of Theorem 3. By Lemma 7, there exists a partition of the vertices of $G-H$ into two parts S and T such that each vertex in S has at least half its neighbors in T, and each vertex in T has at least half its neighbors in S.

Let B be the bipartite subgraph of $G-H$ consisting of the edges that are between S and T, and let M be a maximum matching in B. Let A be the subset of S containing those vertices that are unmatched by M. If $A = \emptyset$, then define $C = D = \emptyset$. Otherwise, consider the set of vertices that are reachable from A by an M-alternating path. Let C be the subset of S that is reachable in this way, and D the subset of T that is reachable in this way. Note that $A \subseteq C$.

By the maximality of M, there is no M-augmenting path in B and so all vertices in D are matched by M. Furthermore, by the construction, M matches each vertex in D with a vertex in C. It follows that

$$|M| = |D \cup (S \setminus C)|.$$

Note that by the construction, there are no edges, in $G-H$, between C and $T \setminus D$. Thus, for any vertex in C, at least half its neighbors are in D. Similarly, for any vertex in $T \setminus D$, at least half its neighbors are in $S \setminus C$.

Let

$$F = D \cup (S \setminus C) \cup H.$$

We claim that F is a k-dominating set for G. For, consider any vertex v that is not in F. As v is not in F, it is not in H either, and so has degree at least $2k-1$. At least half of the neighbors of v are in F, since any neighbor of v that belongs to H is in F, and at least half of the remaining neighbors are in F. It follows that v has at least k neighbors in F.

Thus,

$$\gamma_k(G) \leq |M| + |H| \leq \alpha'(G-H) + |H|.$$

□

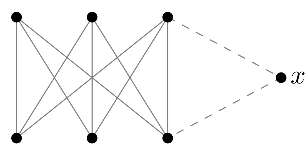

Figure 1: The graph F, for $k = 2$.

3. Proof of Theorem 5

Proof of Theorem 5. By the assumption, the set of vertices of degree less than $2k-2$ is an independent set. Let I be a maximal independent set in $G[H]$ containing all vertices of G of degree less than $2k-2$, and let $J = G - I$. Our strategy will be to construct a k-dominating set of G by taking the union of I and a minimum k-dominating set of a graph obtained by augmenting J in a certain way so that we may appeal to Corollary 2.

We will use the complete bipartite graph $F = K_{2k-1,2k-1}$ to form a graph J^* with $\delta(J^*) \geq 2k-1$ in the following manner. For each vertex x of J of degree less than $2k-1$: introduce $\lceil ((2k-1) - d_J(x))/2 \rceil$ copies of F and attach each to x by two edges such that the ends of the edges are adjacent. See Figure 1.

Let D^* be a minimum k-dominating set of J^*. We claim that D^* contains at least $2k-1$ vertices from each attached F. For, if D^* has less than k vertices from one partite set of the F, then it must have every vertex from the other partite set except possibly the vertex w attached to x; and if vertex $w \notin D^*$ then at least $k-1$ vertices from the other partite set must be in D^*.

Also, we claim that we can choose D^* so that it has exactly $2k-1$ vertices from each attached F. For, if it has more, these can be re-arranged to be one partite set and x. Further, by considering all the possibilities, it follows that an x attached to a F has exactly one neighbor in that F that is in D^*.

Since $\delta(J^*) \geq 2k-1$, we know from Corollary 2 that $|D^*| \leq |V(J^*)|/2$. Set $D = I \cup (D^* \cap V(J))$. From the above it follows that

$$|D| \leq |I| + \frac{|V(J)|}{2} = \frac{n + |I|}{2} \leq \frac{n + \alpha(G[H])}{2}. \tag{1}$$

It remains only to show that D is a k-dominating set of G.

Note that all vertices of J that had no F graphs attached are k-dominated by D. So, let x be a vertex which had at least one of the F graph attached. If $x \in D$ then there is no problem; so assume $x \notin D$.

By the choice of I, $d_G(x) \geq 2k-2$. By the definition of J, vertex x has $d_G(x) - d_J(x)$ neighbors in I. If $d_G(x) \geq 2k-1$, then x has at most $\lceil (d_G(x) - d_J(x))/2 \rceil$ neighbors in $D^* \setminus D$. Since x is k-dominated by D^*, it has at least $k - \lceil (d_G(x) - d_J(x))/2 \rceil$ neighbors in J, and therefore at least k neighbors in D. That is, x is k-dominated by D.

So assume $d_G(x) = 2k-2$. Then $d_J(x) < 2k-2$, since otherwise we contradict the maximality of I. It follows again that vertex x has at least as many neighbors in I as it has in $D^* \setminus D$, and so is k-dominated by D.

Consequently, D is a k-dominating set of G, and together with (1), this completes the proof. □

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