# INDEPENDENCE, RADIUS AND HAMILTONIAN PATHS

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Abstract. We show that if the radius of a simple, connected graph equals its independence number, then the graph contains a Hamiltonian path. This result was conjectured by the computer program Graffiti.pc, using a new conjecture-generating strategy called Sophie. We also mention several other sufficient conditions for Hamiltonian paths that were conjectured by Graffiti.pc, but which are currently open, so far as we know.

## Introduction and Key Definitions

We limit our discussion to graphs that are simple, connected and finite of order $n$. Although we often identify a graph $G$ with its set of vertices, in cases where we need to be explicit we write $V(G)$. We let $\alpha = \alpha(G)$ denote the *independence number* of $G$; this is the maximum order of an induced discrete subgraph of $G$. The *eccentricity* of a vertex $v$ of $G$ is the maximum of the distances from $v$ to the other vertices of $G$. The minimum eccentricity taken over all vertices of $G$ is called the *radius* of $G$ and is denoted by $r = r(G)$. The *path covering number* of $G$ is denoted by $\rho = \rho(G)$ and is the minimum number of vertex-disjoint paths needed to cover the vertices of $G$ (e.g. when $\rho = 1$, $G$ contains a Hamiltonian path). We define the *path number* of $G$, denoted by $p = p(G)$, as the maximum order of an induced path in the graph. One can make an analogous definition for the *bipartite number* of $G$, denoted by $b = b(G)$. Other more specialized definitions will be introduced immediately prior to their first appearance. Standard graph theoretical terms not defined in this paper can be found in [14].

In a classical 1986 paper by P. Erdös, M. Saks, and V. Sós [8], using a proof credited to F. Chung, it is shown that every graph of radius $r$ has an induced path of order at least $2r - 1$. We state this result as Theorem 1, which is sometimes called the Induced Path Theorem [11].

**Theorem 1.** [8] *Let $G$ be a graph. Then*

$$p \geq 2r - 1.$$

Two immediate corollaries of Theorem 1 are summarized in the following Theorem 2. Only the second inequality requires a proof; there are various other proofs of this

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inequality besides the one given here. The best known of these inequalities is the first one: the independence number of a graph is at least as large as its radius. This result was proven independently at roughly the same time as Theorem 1 by S. Fajtlowicz and B. Waller [10], motivated as an early conjecture of the computer program Graffiti [9], as well as by O. Favaron, M. Mahéo and J-F. Saclé [12]. Neither of the these independent proofs is similar to Chung's proof of the Induced Path Theorem.

**Theorem 2.** *Let $G$ be a graph. Then $\alpha \geq r$ and $b \geq 2r$.*

*Proof.* The first inequality is an obvious consequence of Theorem 1. To show $b \geq 2r$, suppose $G$ is a counterexample. Let $P$ be an induced path of order at least $2r - 1$. Now $P$ must have order exactly $2r - 1$ and $b = 2r - 1$, or we are finished. Color the vertices of $P$ red and green. So the endpoints of $P$ have the same color. But each vertex $v$ outside of $P$ must be adjacent to both a red and green vertex of $P$, or $b \geq 2r$ and $G$ is not a counterexample. Thus $v$ must be adjacent to an interior vertex of $P$. But this implies the radius of $G$ is at most $r - 1$, again a contradiction. □

Although it is easy to find graphs (other than cliques) for which these two inequalities are best possible, the problem of characterizing the case of equality for each lower bound has apparently remained unresolved. Of particular interest has been characterizing those graphs where $\alpha = r$ (see [11], [12]). The main goal of this paper is therefore to prove the following Theorem 3, which sheds some light on the structure of these extremal graphs as well as supplying a new sufficient condition for a graph to contain a Hamiltonian path. We defer the proof of this theorem to the next section.

**Theorem 3.** *(Main Theorem) Let $G$ be a graph such that $\alpha = r$. Then $G$ contains a Hamiltonian path.*

One interesting aspect of this theorem is that it applies to various families of graphs, such as even paths and cycles, for which many of the classical sufficient conditions for Hamiltonian paths do not apply. Let us discuss the genesis of this theorem. Graffiti, a computer program that makes conjectures, was written by S. Fajtlowicz and dates from the mid-1980's. Graffiti.pc, a program that makes graph-theoretical conjectures utilizing conjecture-making strategies similar to those found in Graffiti, was written by E. DeLaViña. The operation of Graffiti.pc and its similarities to Graffiti are described in [2] and [3]; its conjectures can be found in [5]. A numbered, annotated listing of several hundred of Graffiti's conjectures can be found in [9]. Both Graffiti and Graffiti.pc have correctly conjectured a number of new bounds for several well studied graph invariants; bibliographical information on resulting papers can be found in [4].

Graffiti.pc employs two main strategies for generating conjectures. The first of these is known as the "Dalmatian heuristic" (due to Fajtlowicz) and generates necessary conditions for a particular class of graphs $P$, as chosen by the user (frequently, this class is merely all simple, connected graphs). Dalmatian conjectures are of the form

*If a graph belongs to class $P$, then "fixed expression $\geq$ expression 2"*

where the expressions on the left and right are composed of graph invariants and constants combined by algebraic operations. The fixed expression on the left is also chosen by the user, and may consist of just a single graph invariant. At present, Graffiti.pc can compute about 500 invariants and 25 operators. Graffiti.pc generates expressions of 35 various types (as determined by the arity of the operators in the expression). Expressions may contain several terms. Different expressions are generated by varying terms and operators over the invariant set and available operators of the appropriate arity, respectively.

Recently, the authors have been experimenting with a new strategy for generating conjectures, called the "Sophie heuristic" (due to DeLaViña and Waller). Graffiti.pc's Sophie heuristic generates sufficient conditions for a particular class of graphs $P$, as chosen by the user. Sophie conjectures are of the form

*If "expression 1 $\geq$ expression 2" for a graph, then the graph belongs to class $P$.*

Sophie generates conjectures by utilizing two databases of graphs and their computed invariants. The *background* database comprises about two million small connected graphs (most generated by B. McKay's geng program), and the *top* database is a small subset of the background database. Let $P$ be a class of graphs. The *target set* is the collection of graphs in the top database that belong to class $P$. The *cover set* $C$ of a relation between expressions is the set of graphs in the top database for which the relation is true. The Sophie heuristic begins by generating pairs of different expressions by varying terms and operators over the invariant set and available operators of the appropriate arity, respectively. Then the cover sets of each of three possible relations (utilizing $\leq, \geq$, and $=$) between a pair of expressions is determined. If the cover set of a relation is contained in the target set, then the relation is considered to be the hypothesis of a candidate conjecture, whose conclusion is that a graph satisfying the hypothesis belongs to class $P$. A candidate conjecture is accepted if its cover set includes graphs not included in any of the cover sets of previously accepted conjectures, and if it is plausible versus the background database. Sophie's goal is to generate a "minimal" list of conjectures that "covers" all graphs in the target set. If this goal is met, then Sophie tries to extend (i.e. add graphs to) the target set and top database, and continues toward the goal.

As one of our initial test beds for Sophie, we chose the class $P$ of simple, connected graphs containing a Hamiltonian path. This test resulted in a collection of 34 conjectures, several of which have now been either proven or refuted. We will mention a few of our favorite open conjectures from this list in the last section. The full list of Sophie conjectures is available at [5]. The conjecture that resulted in Theorem 3 was contained on an early list of Sophie conjectures, but eventually was replaced by the following more general conjecture.

**Conjecture 1.** *(Graffiti.pc* 196*) Let $G$ be a graph. If $b = 2r$, then $G$ contains a Hamiltonian path.*

This conjecture is a generalization of Theorem 3 because since $\alpha \geq b/2$, Theorem 2 implies if $\alpha = r$, then $b = 2r$ as well. Some time after Sophie generated these conjectures, we noticed that Theorem 3 is also a corollary of the following open conjecture of Graffiti.

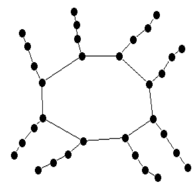

FIGURE 1. The 7-ciliate $C(8, 3)$

**Conjecture 2.** *(Graffiti [7]) Let $G$ be a graph. Then*

$$\alpha \geq r + \frac{\rho - 1}{2}.$$

There exist at least two different generalizations of Theorem 1, provided independently by Fajtlowicz in ([11], Theorem 2), and G. Bacsó and Z. Tuza in ([1], Theorem 1). The 1988 result of Fajtlowicz plays a key role in the proof of Theorem 3. Fajtlowicz proves this result in the context of characterizing *radius-critical* graphs, which are graphs in which every proper induced connected subgraph has radius strictly less than the parent graph. Let $P(n)$ and $C(n)$ denote the path on $n$ vertices and the cycle on $n$ vertices, respectively. Let $C(p, q)$ denote the graph obtained from $p$ disjoint copies of $P(q+1)$ by linking together one endpoint of each path in a cycle $C(p)$. For $1 \leq t \leq r$, the graphs $C(2t, r - t)$ have radius $r$ and are referred to as $r$-ciliates. Ciliates include the even paths $P(2r)$ and even cycles $C(2r)$ as the extreme cases $t = 1$ and $t = r$ (assuming $C(2) = P(2)$). Figure 1 depicts the 7-ciliate $C(8, 3) = C(2 \cdot 4, 7 - 4)$.

**Theorem 4.** *(Fajtlowicz [11]) Let $G$ be a graph with $r \geq 1$. Then $G$ contains an $r$-ciliate as an induced subgraph.*

Finally, another result of Fajtlowicz (also conjectured by Graffiti) will allow us to somewhat simplify the proof of Theorem 3.

**Theorem 5.** *(Fajtlowicz [9]) Let $G$ be a graph with $\alpha = 2$. Then $G$ contains a Hamiltonian path.*

## PROOF OF MAIN THEOREM

**Theorem 3.** *Let $G$ be a graph such that $\alpha = r$. Then $G$ contains a Hamiltonian path.*

*Proof.* The case $\alpha = r = 1$ is trivial. Thus, Theorem 5 implies we can limit our attention to the case $\alpha = r \geq 3$. (We should note that Fajtlowicz has communicated to us a short, independent proof of the case $\alpha = r = 3$.) The structure of $r$-ciliates and Theorem 4 imply the following Lemma 1. Lemma proofs are given in the next section.

**Lemma 1.** *Let $G$ be a graph with $r \geq 1$ such that $\alpha = r$. Then $G$ contains either $P(2r)$ or $C(2r)$ as an induced subgraph. Moreover, if we let $H$ denote an induced $P(2r)$ or $C(2r)$ subgraph, then every vertex of $G$ is either contained in $H$ or is adjacent to $H$.*

**Lemma 2.** *Let $G$ be a graph such that $\alpha = r \geq 1$. Then for each vertex $v$ such that $v \in V(G) - V(H)$, $v$ is adjacent to at least two vertices in $H$.*

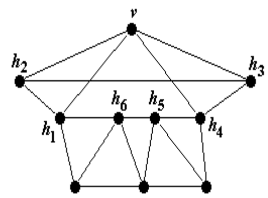


FIGURE 2. $\alpha = r = 3$

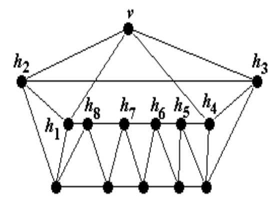


FIGURE 3. $\alpha = r = 4$

Enumerate the vertices of $H$ as $h_1, h_2, h_3, \ldots, h_{2r}$; clockwise if $H$ is a cycle, and left-to-right if $H$ is a path. Let $h_i$ and $h_j$ be two distinct vertices on $H$. Then we define $\delta(h_i, h_j) = \min\{|j-i|, 2r - |j-i|\}$. (Note that if $H$ is a cycle, then $\delta(h_i, h_j)$ is just the shortest-path distance between $h_i$ and $h_j$ with respect to $H$. If $H$ is a path, imagine the cycle $F$ formed from $H$ by joining $h_1$ and $h_{2r}$. Then $\delta(h_i, h_j)$ is just the shortest-path distance between $h_i$ and $h_j$ with respect to $F$.) Moreover, we say that $h_i$ and $h_j$ are *consecutive* provided $\delta(h_i, h_j) = 1$. (Hence, $h_1$ and $h_{2r}$ are consecutive.) Now suppose $v$ is a vertex such that $v \in V(G) - V(H)$. Then we let $\delta(v) = \max\{\delta(h_i, h_j) : v \text{ is adjacent to } h_i, h_j\}$. We have that $\delta(v)$ is well-defined by Lemma 2.

In addition to assuming $\alpha = r$, if we assume $r \geq 5$, then we can show the following Lemma 3.

**Lemma 3.** *Let $G$ be a graph with $r \geq 5$ such that $\alpha = r$. If $H = C(2r)$, then for each vertex $v$ such that $v \in V(G) - V(H)$, $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$.*

The reader may be curious about the necessity of the condition $r \geq 5$ in the statement of Lemma 3. The graphs in Figures 2 and 3 show that for small values of $r$, Lemma 3 may not hold. Each graph contains an induced $C(2r)$ subgraph and a vertex $v$ not on this cycle adjacent to four vertices of the cycle.

**Lemma 4.** *Let $G$ be a graph with $3 \leq r \leq 4$ such that $\alpha = r$. Then either:*

*1) $G$ contains $H = C(2r)$ as an induced subgraph, and for each vertex $v$ such that $v \in V(G) - V(H)$, $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$, or*

*2) $G$ contains $P(2r)$ as an induced subgraph.*

The following sequence of lemmas and definitions culminates in Lemma 14, which then allows us to state an algorithm for constructing a Hamiltonian path in a graph $G$ where $\alpha = r \geq 3$. Lemmas 11, 12, and 13 are analogous to Lemma 3 in the case when $H = P(2r)$.

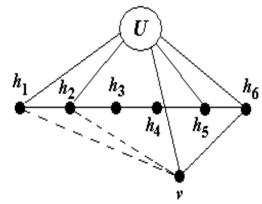


FIGURE 4. Lemma 7

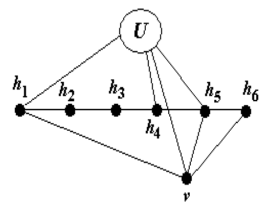


FIGURE 5. Lemma 8

Unlike when $r \geq 5$ and $H = C(2r)$, vertices not in $H$ may be adjacent to up to four vertices in $H$ when $H = P(2r)$. Certain complications arise when $r = 3$, or when there exist vertices not in $H$ that are adjacent to the endpoints of $H$. These complications necessitate a number of mostly technical lemmas, in particular Lemmas 6 through 9.

**Lemma 5.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. If $H = P(2r)$ and $v$ is a vertex such that $v \in V(G) - V(H)$, then $1 \leq \delta(v) \leq 3$.*

**Lemma 6.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Let $v$ be a vertex such that $v \in V(G) - V(H)$. Then the neighbors of $v$ in $H$ must be a subset of four consecutive vertices. Moreover, if $r \geq 4$ and $\delta(v) = 3$, the neighbors of $v$ in $H$ cannot be a subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, $\{h_1, h_2, h_{2r-1}, h_{2r}\}$, or $\{h_1, h_2, h_3, h_{2r}\}$.*

**Lemma 7.** *Let $G$ be a graph with $\alpha = r = 3$. Suppose $H = P(6)$. Let $U \subset V(G) - V(H)$ be a collection of vertices such that for every $u \in U$, $\delta(u) = 3$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_5, h_6\}$. Then $U$ must induce a clique in $G$, and each vertex $u \in U$ must be adjacent to each of the vertices $\{h_1, h_2, h_5, h_6\}$. Moreover, if there exists a vertex $v \in V(G) - V(H)$ adjacent to $h_6$ such that the neighbors of $v$ in $H$ are a subset of $\{h_1, h_2, h_6\}$, then $v$ is adjacent to each vertex of $U$. (See Figure 4.)*

**Lemma 8.** *Let $G$ be a graph with $\alpha = r = 3$. Suppose $H = P(6)$. Let $U \subset V(G) - V(H)$ be a collection of vertices such that for every $u \in U$, $\delta(u) = 3$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_4, h_5, h_6\}$. Then $U$ must induce a clique in $G$, and each vertex $u \in U$ must be adjacent to exactly the set $\{h_1, h_4, h_5\}$ in $H$. Moreover, there exists a vertex $v \in V(G) - V(H)$ adjacent to exactly the set $\{h_1, h_5, h_6\}$ in $H$, and each such vertex $v$ is adjacent to each vertex of $U$. (See Figure 5.)*

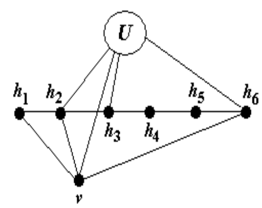


FIGURE 6. Lemma 9

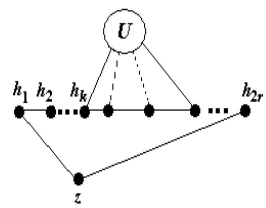


FIGURE 7. Lemma 10.1)

**Lemma 9.** *Let $G$ be a graph with $\alpha = r = 3$. Suppose $H = P(6)$. Let $U \subset V(G) - V(H)$ be a collection of vertices such that for every $u \in U$, $\delta(u) = 3$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_3, h_6\}$. Then $U$ must induce a clique in $G$, and each vertex $u \in U$ must be adjacent to exactly the set $\{h_2, h_3, h_6\}$ in $H$. Moreover, there exists a vertex $v \in V(G) - V(H)$ adjacent to exactly the set $\{h_1, h_2, h_6\}$ in $H$, and each such vertex $v$ is adjacent to each vertex of $U$. (See Figure 6.)*

**Lemma 10.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Assume $H = P(2r)$. Suppose $U$ is a collection of vertices such that $U \subset V(G) - V(H)$ and $k = min\{j : u \in U$ and $u$ is adjacent to $h_j\}$. Moreover, suppose for every $u \in U$ that $u$ is adjacent to $h_k$, and $\delta(v) = 3$ for some $v \in U$. Then:*

*1) If $2 \leq k \leq 2r - 4$, then there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to both $h_1$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these two vertices in $H$, and $z$ is not adjacent to any vertex $u \in U$. (See Figure 7.)*

*2) If $k = 1$ and for every vertex $u \in U$, $u$ is adjacent to $h_4$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_3, h_4\}$, then there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_1$ and at least one of $h_2$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is not adjacent to any vertex $u \in U$. (See Figure 8.)*

*3) If $k = 2r - 3$ and for every vertex $u \in U$, the neighbors of $u$ in $H$ are a subset of $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$, then there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_{2r}$ and at least one of $h_1$ or $h_{2r-1}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is not adjacent to any vertex $u \in U$. (See Figure 9.)*

**Lemma 11.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Moreover, suppose $v$ is a vertex such that $v \in V(G) - V(H)$ and the neighbors of $v$ include neither $h_1$ nor $h_{2r}$. Then $v$ is adjacent to exactly two, exactly three, or exactly four consecutive vertices in $H$.*

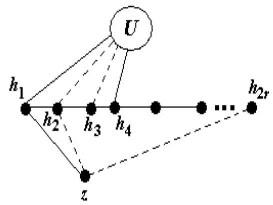


FIGURE 8. Lemma 10.2)

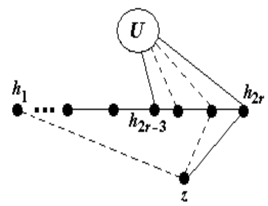


FIGURE 9. Lemma 10.3)

**Lemma 12.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Moreover, suppose $v$ is a vertex such that $v \in V(G) - V(H)$ and the neighbors of $v$ include $h_1$. Then either:*
*1) $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$; or*
*2) $v$ is adjacent to exactly $h_1$, $h_2$, $h_3$, and $h_4$ in $H$; or*
*3) $v$ is adjacent to $h_1$, $h_3$, and $h_4$; or*
*4) $r = 3$ and $v$ is adjacent to exactly $h_1$, $h_2$, $h_5$, and $h_6$ in $H$; or*
*5) $r = 3$ and $v$ is adjacent to exactly $h_1$, $h_4$, and $h_5$ in $H$.*

**Lemma 13.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Moreover, suppose $v$ is a vertex such that $v \in V(G) - V(H)$ and the neighbors of $v$ include $h_{2r}$. Then either:*
*1) $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$; or*
*2) $v$ is adjacent to exactly $h_{2r-3}$, $h_{2r-2}$, $h_{2r-1}$, and $h_{2r}$ in $H$; or*
*3) $v$ is adjacent to exactly $h_{2r-3}$, $h_{2r-2}$, and $h_{2r}$ in $H$; or*
*4) $r = 3$ and $v$ is adjacent to exactly $h_1$, $h_2$, $h_5$, and $h_6$ in $H$; or*
*5) $r = 3$ and $v$ is adjacent to exactly $h_2$, $h_3$, and $h_6$ in $H$.*

Let $G$ be a graph with $r \geq 1$. Suppose $G$ contains an induced subgraph $H$ such that $H = P(2r)$ or $H = C(2r)$. Suppose $u$ and $v$ are a pair of vertices where $u, v \in V(G) - V(H)$. Let $k$ be the smallest integer such that $u$ is adjacent to $h_k$ and $k'$ be the smallest integer such that $v$ is adjacent to $h_{k'}$. Then $u$ and $v$ are said to be *degenerate* (with respect to $H$) if either:
a) the union of their neighbors in $H$ is three or less consecutive vertices; or
b) $k = k'$, and the union of their neighbors in $H$ is four or less consecutive vertices including neither $h_1$ nor $h_{2r}$; or

c) $k = k' = 1$, both are adjacent to $h_4$, and the union of their neighbors in $H$ is a subset of $\{h_1, h_2, h_3, h_4\}$; or

d) $k = k' = 2r - 3$, either $\delta(u) = 3$ or $\delta(v) = 3$, and the union of their neighbors in $H$ is a subset of $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$; or

e) their neighbors in $H$ are identical.

**Lemma 14.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Then there exists a subgraph $H$ of $G$ such that either $H = P(2r)$ or $H = C(2r)$, and if $u, v \in V(G) - V(H)$ is a pair of degenerate vertices with respect to $H$, $u$ is adjacent to $v$.*

We can now complete the proof of the main theorem. Let us repeat the choice of the induced subgraph $H$ as described in the proof of Lemma 14 (below). If $r \geq 5$ and $G$ contains an induced $C(2r)$ subgraph, let $H$ be this subgraph. Then $G$ and $H$ satisfy Lemma 3. If $r \geq 5$ and $G$ does not contain an induced $C(2r)$ subgraph, let $H$ be the induced $P(2r)$ subgraph implied by Lemma 1. If $3 \leq r \leq 4$ and $G$ contains an induced $C(2r)$ subgraph that satisfies Lemma 4, let $H$ be this subgraph. If $3 \leq r \leq 4$ and $G$ does not contain an induced $C(2r)$ subgraph that satisfies Lemma 4, let $H$ be the induced $P(2r)$ subgraph implied by Lemma 4.

For each $k$, $1 \leq k \leq 2r - 3$, let $X_k$ denote the set of vertices in $V(G) - V(H)$ that are adjacent to $h_k$ but whose neighbors in $H$ are a subset of $\{h_k, h_{k+1}, h_{k+2}, h_{k+3}\}$. Moreover, let $X_{2r-2}$ denote the set of vertices in $V(G) - V(H)$ that are adjacent to $h_{2r-2}$ but whose neighbors in $H$ are a subset of $\{h_{2r-2}, h_{2r-1}, h_{2r}\}$; let $X_{2r-1}$ denote the set of vertices in $V(G) - V(H)$ that are adjacent to $h_{2r-1}$ but whose neighbors in $H$ are a subset of $\{h_{2r-1}, h_{2r}, h_1\}$; and let $X_{2r}$ denote the set of vertices in $V(G) - V(H)$ that are adjacent to $h_{2r}$ but whose neighbors in $H$ are a subset of $\{h_{2r}, h_1, h_2\}$. There are three exceptions to this scheme, which are necessary only when $r = 3$ and $H = P(6)$. We also include in $X_{2r-1}$ those vertices in $V(G) - V(H)$ that are adjacent to exactly $h_{2r-2}$, $h_{2r-1}$, and $h_1$ in $H$. Likewise, we include in $X_{2r}$ those vertices in $V(G) - V(H)$ that are adjacent to exactly $h_{2r-1}$, $h_{2r}$, $h_1$, and $h_2$ in $H$, as well as those vertices that are adjacent to exactly $h_2$, $h_3$, and $h_{2r}$ in $H$. By Lemmas 3, 4, 11, 12, and 13, each vertex of $V(G) - V(H)$ is contained in precisely one of the sets $X_j$. By Lemma 14, each set $X_j$ induces a clique in $G$, except possibly: $X_1$ when $H = P(2r)$; $X_{2r-1}$ when $r = 3$ and $H = P(6)$; and $X_{2r}$ when $r = 3$ and $H = P(6)$. We inductively construct a Hamiltonian path $P$.

Step 1) Consider first the case $H = C(2r)$. Let $h_1$ be the initial vertex of $P$. If $X_1$ is empty, then we add $h_2$ to $P$ and proceed. Otherwise, assume there exists a vertex $v \in X_1$, which by definition must be adjacent to $h_1$. We next add $v$ to $P$, and because $X_1$ induces a clique in $G$ in this case, we can in turn add each additional vertex of $X_1$ to $P$ as well. Assume $u$ is the last vertex of $X_1$ added to $P$ in this fashion. By our choice of $H$ and Lemma 3, $u$ must be adjacent to $h_2$. Hence we can add $h_2$ to $P$ and continue.

Next, consider the case $H = P(2r)$. If $X_1$ is empty, let $h_1$ be the initial vertex of $P$. Then we add $h_2$ to $P$ and proceed. Otherwise, let $Y$ be those vertices $y \in X_1$ such that $\delta(y) = 3$, and let $Z$ be those vertices $z \in X_1$ such that $\delta(z) \leq 2$. By Lemma 12, $Y \cup Z = X_1$. If $Y$ is empty, let $h_1$ be the initial vertex of $P$. Otherwise, assume there exists a vertex $y \in Y$. Let $y$ be the initial vertex of $P$, and because $Y$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Y$ to $P$ as well. Assume $u$ is the last vertex of $Y$ added to $P$ in this fashion. Since $Y \subset X_1$, $u$ must be adjacent to

$h_1$. Hence we can now add $h_1$ to $P$. In any event, $h_1$ is added to $P$. Next, if $Z$ is empty, then we add $h_2$ to $P$ and proceed. Otherwise, assume there exists a vertex $z \in Z \subset X_1$. Add $z$ to $P$, and because $Z$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Z$ to $P$ as well. Assume $w$ is the last vertex of $Z$ added to $P$ in this fashion. Since $\delta(w) \leq 2$, $w$ must be adjacent to $h_2$ by Lemma 12. Hence we can add $h_2$ to $P$ and continue.

Step 2) Now suppose we have constructed a path $P$ such that the terminal vertex of $P$ is $h_j$ $(1 < j < 2r-1)$; each vertex $h_1, h_2, \ldots, h_{j-1}$ is contained in $P$; and each vertex of $X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_{j-1}$ is contained in $P$. Moreover, suppose these are the only vertices contained in $P$. If $X_j$ is empty, then we add $h_{j+1}$ to $P$ and continue. Otherwise, assume there exists a vertex $v \in X_j$, which by definition must be adjacent to $h_j$. We next add $v$ to $P$ ($v \notin X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_{j-1}$ assures that $v$ is not already contained on $P$). Because $X_j$ induces a clique in $G$, we can in turn add each additional vertex of $X_j$ to $P$. Assume $u$ is the last vertex of $X_j$ added to $P$ in this fashion. By Lemmas 3, 4, 11, and 13, $u$ must be adjacent to $h_{j+1}$. Hence we can add $h_{j+1}$ to $P$ and continue.

Step 3) Now suppose we have constructed a path $P$ such that the terminal vertex of $P$ is $h_{2r-1}$; each vertex $h_1, h_2, \ldots, h_{2r-2}$ is contained in $P$; and each vertex of $X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_{2r-2}$ is contained in $P$. Moreover, suppose these are the only vertices contained in $P$. If $X_{2r-1}$ is empty, then we add $h_{2r}$ to $P$ and continue. Otherwise, assume there exists a vertex $v \in X_{2r-1}$, which by definition must be adjacent to $h_{2r-1}$. Consider first the cases $H = C(2r)$, or $r \geq 4$ and $H = P(2r)$. We next add $v$ to $P$, and because $X_{2r-1}$ induces a clique in $G$ in these cases by Lemma 14, we can in turn add each additional vertex of $X_{2r-1}$ to $P$ as well. Assume $u$ is the last vertex of $X_{2r-1}$ added to $P$ in this fashion. By our choice of $H$ and Lemmas 3, 11, and 13, $u$ must be adjacent to $h_{2r}$. Hence we can add $h_{2r}$ to $P$ and continue.

Next, consider the case $r = 3$ and $H = P(6)$. If $X_{2r-1}$ is empty, then we add $h_{2r}$ to $P$ and proceed. Otherwise, let $Y$ be those vertices in $X_{2r-1}$ that are adjacent to exactly $h_4$, $h_5$, and $h_1$ in $H$, and let $Z$ be those vertices in $X_{2r-1}$ such that $\delta(z) \leq 2$. By Lemmas 11, 12, and 13, $Y \cup Z = X_{2r-1}$. If $Y$ is not empty, assume $y \in Y$. By Lemma 8, there exists $z \in Z$ also. Add $y$ to $P$, and because $Y$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Y$ to $P$ as well. Assume $u$ is the last vertex of $Y$ added to $P$ in this fashion. By Lemma 8, $u$ must be adjacent to $z$. Hence we can now add $z$ to $P$. If $Y$ is empty, then $Z$ is not empty, in which case by the definition of $X_{2r-1}$, we can add $z$ to $P$. In any event, $z$ is added to $P$. Because $Z$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Z$ to $P$ as well. Assume $w$ is the last vertex of $Z$ added to $P$ in this fashion. Since $\delta(w) \leq 2$, $w$ must be adjacent to $h_{2r}$ by Lemmas 11, 12, and 13. Hence we can add $h_{2r}$ to $P$ and continue.

Step 4) Now suppose we have constructed a path $P$ such that the terminal vertex of $P$ is $h_{2r}$; each vertex $h_1, h_2, \ldots, h_{2r-1}$ is contained in $P$; and each vertex of $X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_{2r-1}$ is contained in $P$. Moreover, suppose these are the only vertices contained in $P$. If $X_{2r}$ is empty, then we are finished. Otherwise, assume there exists a vertex $v \in X_{2r}$. Consider first the cases $H = C(2r)$, or $r \geq 4$ and $H = P(2r)$. By definition, $v$ must be adjacent to $h_{2r}$ in these cases. We next add $v$ to $P$, and because $X_{2r}$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $X_{2r}$ to $P$ as well. The theorem now follows.

Next, consider the case $r = 3$ and $H = P(6)$. If $X_{2r}$ is empty, then we add $h_{2r}$ to $P$ and are finished as before. Otherwise, let $Y$ be those vertices in $X_{2r}$ that are adjacent to exactly $h_1$, $h_2$, $h_5$, and $h_6$ in $H$; let $Z$ be those vertices in $X_{2r}$ such that $\delta(z) \leq 2$; and let $A$ be those vertices in $X_{2r}$ that are adjacent to exactly $h_2$, $h_3$, and $h_6$ in $H$. By Lemmas 11, 12, and 13, $Y \cup Z \cup A = X_{2r}$. We consider three cases.

*Case 1: $A$ is empty.* If $Y$ is not empty, assume $y \in Y$. Add $y$ to $P$, and because $Y$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Y$ to $P$ as well. Assume $u$ is the last vertex of $Y$ added to $P$ in this fashion. If $Z$ is empty, then we are finished. Otherwise, suppose $z \in Z$. By Lemma 7, $u$ must be adjacent to $z$. Hence we can now add $z$ to $P$. If $Y$ is empty, then $Z$ is not empty, in which case by the definition of $X_{2r-1}$, we can add $z$ to $P$. In any event, $z$ is added to $P$. Because $Z$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Z$ to $P$ as well, and are finished.

*Case 2: $A$ is not empty, but $Y$ is empty.* By Lemma 9, $Z$ cannot be empty, and moreover, there exists $z \in Z$ such that $z$ is adjacent to each vertex of $A$. If $Z$ contains at least two vertices, let $z' \in Z$ be some vertex other than $z$. If $Z$ consists of only $z$, let $z' = z$. By definition, $z'$ must be adjacent to $h_{2r}$. Hence we can now add $z'$ to $P$, and because $Z$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Z$ to $P$ in such a way that $z$ is the last vertex added to $P$ in this fashion. Now assume $a \in A$. Since $a$ is adjacent to $z$, we can next add $a$ to $P$. Since $A$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $A$ to $P$, and are finished.

*Case 3: $A$ is not empty, and $Y$ is not empty.* By Lemma 9, $Z$ cannot be empty, and moreover, there exists $z \in Z$ such that $z$ is adjacent to each vertex of $A$. If $Z$ contains at least two vertices, let $z' \in Z$ be some vertex other than $z$. If $Z$ consists of only $z$, let $z' = z$. Assume $y \in Y$ and $a \in A$. Add $y$ to $P$, and because $Y$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Y$ to $P$ as well. Assume $u$ is the last vertex of $Y$ added to $P$ in this fashion. By Lemma 7, $u$ must be adjacent to $z'$. Hence we can now add $z'$ to $P$, and because $Z$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $Z$ to $P$ in such a way that $z$ is the last vertex added to $P$ in this fashion. Since $a$ is adjacent to $z$, we can next add $a$ to $P$. Since $A$ induces a clique in $G$ by Lemma 14, we can in turn add each additional vertex of $A$ to $P$, and are finished.

The theorem again follows. □

## Proofs of Lemmas

**Lemma 1.** *Let $G$ be a graph with $r \geq 1$ such that $\alpha = r$. Then $G$ contains either $P(2r)$ or $C(2r)$ as an induced subgraph. Moreover, if we let $H$ denote an induced $P(2r)$ or $C(2r)$ subgraph, then every vertex of $G$ is either contained in $H$ or is adjacent to $H$.*

*Proof.* Let $H$ be an induced $r$-ciliate guaranteed by Theorem 4. If $H$ is neither $P(2r)$ nor $C(2r)$, then considering the definition of $r$-ciliates, $\alpha(H) > r$. Since $H$ is induced, $\alpha(G) \geq \alpha(H) > r$, a contradiction. Hence $H = P(2r)$ or $H = C(2r)$. Now suppose $v$ is a vertex of $G$ not contained in $H$. If $v$ is not adjacent to $H$, then clearly we can find an independent set in $G$ including $v$ with order $r+1$, again a contradiction. $\square$

**Lemma 2.** *Let $G$ be a graph such that $\alpha = r \geq 1$. Then for each vertex $v$ such that $v \in V(G) - V(H)$, $v$ is adjacent to at least two vertices in $H$.*

*Proof.* If $v$ is not adjacent to at least two vertices in $H$, then clearly we can find an independent set in $G$ including $v$ with order $r+1$, a contradiction. $\square$

**Lemma 3.** *Let $G$ be a graph with $r \geq 5$ such that $\alpha = r$. If $H = C(2r)$, then for each vertex $v$ such that $v \in V(G) - V(H)$, $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$.*

*Proof.* Let $a$ and $b$ be two neighbors of $v$ in $H$ such that $\delta(v) = \delta(a, b)$. Put $\delta = \delta(v)$. Clearly $\delta \leq r$. First, suppose $\delta \leq 2$. Then $v$ is adjacent to a subset of three consecutive vertices in $H$. If $\delta = 1$, then $v$ is adjacent to two consecutive vertices. But if $\delta = 2$, and $v$ is not adjacent to three consecutive vertices in $H$, then clearly we can find an independent set in $G$ including $v$ with order $r+1$, a contradiction.

Next, by way of contradiction, suppose $\delta \geq 3$. Now $v, a, b$ are contained in $C(\delta+2)$ and $C(2r-\delta+2)$ subgraphs, which share only these three vertices. Let $C_1$ denote the $C(\delta+2)$ subgraph and let $C_2$ denote the $C(2r-\delta+2)$ subgraph. Note that $V(C_1) \cup V(C_2) = V(H) \cup \{v\}$. We consider three cases.

*Case 1:* Suppose $\delta = r$. Then $C_1 = C(r+2)$ and $C_2 = C(r+2)$. Since each vertex $w \in V(G) - V(H)$ is adjacent to at least two vertices in $H$, then the eccentricity of $v$ is at most $\lfloor \frac{r+2}{2} \rfloor + 1$. Because $r \geq 5$, the eccentricity of $v$ is at most $r-1$, a contradiction.

*Case 2:* Suppose $4 \leq \delta \leq r-1$. Now $\delta + 2 \leq r+1$ and $2r - \delta + 2 \leq 2r - 2$. Then the eccentricity of $v$ with relation to $H$ is at most $r-1$, which only occurs when $\delta = 4$. If $4 < \delta \leq r-1$, since each vertex not in $H$ is adjacent to at least two vertices in $H$, then the eccentricity of $v$ is at most $r-1$, a contradiction. Hence $\delta = 4$, which in turn implies there exists a unique vertex $c$ in $H$ at distance $r-1$ from $v$. Since each vertex not in $H$ is adjacent to at least two vertices in $H$, then the eccentricity of $v$ is at most $r-1$, a contradiction.

*Case 3:* Suppose $\delta = 3$. Let $c, d$ be the two vertices separating $a$ from $b$ in $H$. Now $C_1 = C(5)$ and $C_2 = C(2r-1)$. Enumerate the vertices of $C_2$ as $x_0 = v, x_1 = a, x_2, \ldots, x_{2r-2} = b$. For each vertex $x_0, x_1, x_2, \ldots, x_{r-2}$ and $x_{r+1}, x_{r+2}, \ldots, x_{2r-2}$, there exist unique vertices $y_j, z_j$ in $H$ such that the distance from $x_j$ to both $y_j$ and $z_j$ with relation to $C_2$ is $r-1$. Note that $y_j$ and $z_j$ are adjacent. But the distance from $v$ to both $c$ and $d$ is at most 2, and the distance from $v$ to the remaining vertices in $H$ is at most $r-1$. Thus there exists a non-empty collection of vertices $Z_0$ not in $H$ adjacent only to both $y_0 = x_{r-1}$ and $z_0 = x_r$ in $H$, otherwise the eccentricity of $v = x_0$ is at most $r-1$, a contradiction. Recall $r \geq 5$, and consider $x_1$. Then $y_1 = x_{r+1}$ and $z_1 = x_r$. Because the distance from $x_1$ to both $c$ and $d$ is at most $2 \leq r-2$, there exists a non-empty collection of vertices $Z_1$ not in $H$ adjacent only to both $x_r$ and $x_{r+1}$ in $H$; otherwise the eccentricity of $x_1$ is at most $r-1$, a contradiction. Likewise, there exists a non-empty collection of vertices $Z_2$ adjacent only to both $x_{r+1}$ and $x_{r+2}$ in $H$; a non-empty collection of vertices $Z_{2r-2}$ adjacent only to both $x_{r-1}$ and $x_{r-2}$ in $H$; and a non-empty collection of vertices $Z_{2r-3}$

adjacent only to both $x_{r-2}$ and $x_{r-3}$ in $H$. But this implies $\alpha \geq r+1$, unless each vertex of $Z_1$ is adjacent to each vertex of $Z_{2r-3}$. However, then the eccentricity of $x_1$ is at most $r-1$, again a contradiction. □

**Lemma 4.** *Let $G$ be a graph with $3 \leq r \leq 4$ such that $\alpha = r$. Then either:*

*1) $G$ contains $H = C(2r)$ as an induced subgraph, and for each vertex $v$ such that $v \in V(G) - V(H)$, $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$, or*

*2) $G$ contains $P(2r)$ as an induced subgraph.*

*Proof.* Let us suppose $H = C(2r)$. We shall show that if $v$ is not adjacent to exactly two or exactly three consecutive vertices in $H$, then $G$ also contains $P(2r)$ as an induced subgraph. Let $a$ and $b$ be two neighbors of $v$ on $H$ such that $\delta(v) = \delta(a, b)$. Put $\delta = \delta(v)$. Clearly $\delta \leq r$.

First, suppose $\delta \leq 2$. Then $v$ is adjacent to a subset of three consecutive vertices in $H$. If $\delta = 1$, then $v$ is adjacent to two consecutive vertices, a contradiction. But if $\delta = 2$, and $v$ is not adjacent to three consecutive vertices in $H$, then clearly we can find an independent set in $G$ including $v$ with order $r+1$, a contradiction.

Therefore, we can assume $\delta \geq 3$. If $r = 3$, then $\delta = 3$, since $\delta \leq r$. If $r = 4$, then $\delta \leq 4$. Now $v, a, b$ are contained in $C(\delta+2)$ and $C(2r-\delta+2)$ subgraphs, which share only these three vertices. Let $C_1$ denote the $C(\delta+2)$ subgraph and let $C_2$ denote the $C(2r-\delta+2)$ subgraph. Note that $V(C_1) \cup V(C_2) = V(H) \cup \{v\}$. We consider two cases.

*Case 1:* First suppose $r = 4$. If $\delta = 4$, then $C_1 = C(6)$ and $C_2 = C(6)$. Thus the eccentricity of $v$ with relation to $H$ is at most 3. Let $c$ and $d$ be the unique vertices at distance 3 from $v$ with relation to $C_1$ and $C_2$, respectively. Since each vertex not in $H$ is adjacent to at least two vertices in $H$, there must exist a vertex $w$ not in $H$ adjacent to only $c$ and $d$ in $H$, otherwise the eccentricity of $v$ would be 3, a contradiction. However, in this case, we can choose an independent set of size 5 containing $w$, another contradiction. Hence, we can assume $\delta = 3$, $C_1 = C(5)$, and $C_2 = C(7)$. Let $c$ and $d$ be the unique vertices at distance 3 from $v$ with respect $C_2$. Again, since each vertex not in $H$ is adjacent to at least two vertices in $H$, there must exist a vertex $w$ not in $H$ adjacent to only $c$ and $d$ in $H$, otherwise the eccentricity of $v$ would be 3, a contradiction. But now we can find an induced $P(8)$ in $G$, starting with $w$ and including all the vertices of $H$ except $d$.

*Case 2:* Next suppose $r = 3$. Then $\delta = 3$ as noted earlier, $C_1 = C(5)$, and $C_2 = C(5)$. We can assume $a = h_1$ and $b = h_4$.

*Claim:* $v$ is adjacent to either both $h_5$ and $h_6$, or $h_2$ and $h_3$. By way of contradiction, suppose $v$ is adjacent to neither $h_2$ nor $h_6$. Then $\{v, h_2, h_6\}$ is an independent set in $G$ of size 3. Note that the eccentricity of $a = h_1$ with respect to $H$ is at most 2. Moreover, each vertex $w \neq v$ not in $H$ must be adjacent to some vertex in $\{v, h_2, h_6\}$; otherwise $\alpha = 4$, a contradiction. This implies the eccentricity of $a$ is 2 with respect to $G$, a contradiction. Hence, $v$ is adjacent to either $h_2$ or $h_6$. Next, by a symmetrical argument, we have $v$ is adjacent to either $h_3$ or $h_5$.

If $v$ is adjacent to either both $h_5$ and $h_6$, or both $h_2$ and $h_3$, then the claim is established. Thus we can assume $v$ is adjacent to only both $h_3$ and $h_6$ among $\{h_2, h_3, h_5, h_6\}$. Since the eccentricity of $v$ with respect to $H$ is 2, and each vertex not in $H$ is adjacent to at least two vertices in $H$, there must exist a vertex $w$ not in $H$ adjacent to only both $h_2$ and

$h_5$ in $H$. Otherwise, the eccentricity of $v$ with respect to $G$ is at most 2, a contradiction. Now, by letting $w$ play the role of $v$ in the preceding paragraph, we can show $w$ must be adjacent to either $h_1$ or $h_3$, a contradiction. This completes the claim.

In light of the claim, we can assume $v$ is adjacent to both $h_2$ and $h_3$. Once again, since the eccentricity of $v$ with respect to $H$ is at most 2, and each vertex not in $H$ is adjacent to at least two vertices in $H$, there must exist a vertex $w$ not in $H$ adjacent to only both $h_5$ and $h_6$ in $H$. Otherwise, the eccentricity of $v$ with respect to $G$ is at most 2, a contradiction. But now we can find an induced $P(6)$ in $G$, starting with $w$ and including all the vertices of $H$ except $h_6$. □

**Lemma 5.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. If $H = P(2r)$ and $v$ is a vertex such that $v \in V(G) - V(H)$, then $1 \leq \delta(v) \leq 3$.*

*Proof.* Since $v$ must be adjacent to at least two vertices in $H$ from Lemma 2, the lower bound is obvious. Proceeding by contradiction, suppose $\delta(v) \geq 4$ (noting that this assumption implies $r > 3$). Let $h_a$ and $h_b$ be the two center vertices on the even path $H$, where $a < b$. Let $h_m$ and $h_n$ be two vertices in $H$ such that $\delta(v) = \delta(h_m, h_n)$, where we assume $n > m$. We consider two main cases: 1) $m < a$ and $n > b$ (the centers are between the vertices $h_m$ and $h_n$ in $H$); and 2) $m \geq a$ or $n \leq b$.

*Case 1a:* Suppose that $a-m = n-b$. We show that the eccentricity of $v$ is at most $r-1$, a contradiction. First, let $d_F(x, y)$ be the shortest path distance from vertex $x$ to vertex $y$ contained in a graph $F$. Without loss of generality, we can assume $d_G(v, b) \leq d_G(v, a)$, as well as $d_G(v, h_{2r}) \leq d_G(v, h_1)$. Note that there are at least two vertices in $H$ to the left of $h_m$ and at least two vertices in $H$ to the right of $h_n$, because $\delta(v) \geq 4$ and there must be the same number on either side due to our supposition. Consequently,

$d_G(v, h_b) \leq d_G(v, h_a)$
$\leq d_G(h_m, h_a) + 1$
$\leq d_H(h_1, h_a) - d_H(h_1, h_m) + 1$
$\leq d_H(h_1, h_a) - 1$
$= r - 2$

Now, since $\delta(v) \geq 4$ and $a - m = n - b$, we can be assured that $d_H(h_m, h_a) = d_H(h_n, h_b) \geq 2$. So,

$d_G(v, h_{2r}) \leq d_G(v, h_1)$
$\leq d_G(h_1, h_m) + 1$
$\leq d_H(h_1, h_a) - d_H(h_m, h_a) + 1$
$\leq d_H(h_1, h_a) - 1$
$= r - 2$

By the same token, $v$ can reach the other vertices of $H$ in at most $r-2$ steps. This implies its eccentricity is at most $r-1$, since every vertex in $V(G) - V(H)$ is adjacent to a vertex in $H$ by Lemma 1.

*Case 1b:* Suppose that, without loss of generality, $a - m < n - b$. We again show that the eccentricity of $v$ is at most $r-1$, a contradiction. First, observe that since $\delta(v) \geq 4$, there are at most $2r-3$ vertices on the induced subpath of $H$ starting with $h_m$ and ending with $h_n$. This implies the distance between $v$ and every vertex of this induced subpath is at most $r-1$. Furthermore, since $0 < a - m < n - b$, there is at least one vertex strictly between $h_b$ and $h_n$ in $H$. Hence, $d_G(v, h_{2r}) \leq d_H(h_b, h_{2r}) - 1 = r - 2$. Finally,

$d_G(v, h_1) \leq d_H(h_1, h_a) = r-1$, because $d_H(h_m, h_a) \geq 1$. Putting all this together, $v$ can reach every vertex in $H$, except possibly $h_1$, in at most $r-2$ steps. Together with the facts that $v$ can reach $h_1$ in at most $r-1$ steps, and every vertex in $H$ is adjacent to at least one vertex in $H$ other than $h_1$ (from Lemma 2), it follows that the eccentricity of $v$ is at most $r-1$.

*Case 2a:* Suppose that, without loss of generality, $n \leq b$ and $m > 1$. We show that the eccentricity of $h_b$ is at most $r-1$, a contradiction. Since $\delta(v) \geq 4$, $d_G(h_b, h_1) \leq r-2$. Furthermore, since $d_G(h_b, h_i) \leq r-2$ for $3 \leq i \leq b-1 = a$, and $d_G(h_b, h_2) \leq r-3$, we conclude that $h_b$ is at most $r-2$ steps from any vertex in $H$ with index less than $b$. This is also obviously true for all vertices in $H$ with indices at least $b$, excepting only $h_{2r}$, since $d_H(h_b, h_{2r}) = r-1$. As in the prior case, it follows that the eccentricity of $h_b$ is at most $r-1$.

*Case 2b:* Suppose that, without loss of generality, $n \leq b$ and $m = 1$. We show that the eccentricity of $h_b$ is at most $r-1$, or $\alpha \geq r+1$, a contradiction either way. Assume that the eccentricity of $h_b$ is at least $r$. First, observe that $d_G(h_b, h_1) \leq r-2$ (since $\delta(v) \geq 4$), and that $d_G(h_b, h_i) \leq r-2$ for $3 \leq i \leq b-1 = a$. Hence, the only vertices in $H$ which could be at distance $r-1$ from $h_b$ are $h_2$ and $h_{2r}$. Since the eccentricity of $h_b$ is at least $r$, there must exist a vertex $z \in V(G) - V(H)$ which is adjacent to only these two vertices in $H$. But then the vertices in $H$ with odd indices together with $z$ form an independent set of order $r+1$. $\square$

**Lemma 6.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Let $v$ be a vertex such that $v \in V(G) - V(H)$. Then the neighbors of $v$ in $H$ must be a subset of four consecutive vertices. Moreover, if $r \geq 4$ and $\delta(v) = 3$, the neighbors of $v$ in $H$ cannot be a subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, $\{h_1, h_2, h_{2r-1}, h_{2r}\}$, or $\{h_1, h_2, h_3, h_{2r}\}$.*

*Proof.* By Lemma 2, we know that each $v \in V(G) - V(H)$ is adjacent to at least two vertices in $H$. If $v$ has exactly two neighbors in $H$, then its neighborhood restricted to $H$ is clearly a subset of four consecutive vertices, since $\delta(v) \leq 3$. So we assume that $v$ has at least three neighbors in $H$ and consider two cases for $r$.

*Case 1:* Suppose that $r = 3$. If $v$ is not adjacent to either of $h_1$ or $h_6$, then its neighborhood restricted to $H$ is clearly a subset of four consecutive vertices. So without loss of generality suppose that $v$ is adjacent to $h_1$. We now consider two subcases.

*Case 1a:* Assume that $v$ is also adjacent to vertex $h_2$. If $v$ has exactly three neighbors in $H$, then it is easily verified that its neighborhood restricted to $H$ is a subset of four consecutive vertices. So we assume that $v$ has four neighbors in $H$. In this case, there are six possibilities for the adjacencies of the other two neighbors of $v$ in $H$. If the additional two neighbors are exactly $\{h_3, h_4\}$, $\{h_3, h_6\}$ or $\{h_5, h_6\}$, then the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. So to complete the proof of this subcase, we proceed by way of contradiction and assume that the other two neighbors of $v$ in $H$ are $\{h_3, h_5\}$, $\{h_4, h_5\}$, or $\{h_4, h_6\}$. First, assume that $v$ is also adjacent to $h_3$ and $h_5$. It is straightforward to verify that each vertex in $H$ is at distance at most two from $v$, and that vertices $h_4$ and $h_6$ are the only vertices in $H$ at distance two from $v$. Thus there must exist a vertex $z_v$ not in $H$ at distance three from $v$ that is adjacent to both $h_4$ and $h_6$, and no other vertices of $H$. But now $\{z_v, h_1, h_3, h_5\}$ determines an independent set of order four, a contradiction to $\alpha = r = 3$. Assume that $v$ is instead adjacent to $h_4$ and $h_5$. It is straightforward to verify that each vertex in $H$ is at distance at most two from $v$, and that vertices $h_3$ and $h_6$ are the only vertices in $H$ at distance two from $v$. Thus there must exist a vertex $z_v$ not in $H$ at distance three from $v$ that is adjacent to both $h_3$ and $h_6$, and no other vertices of $H$. It is straightforward to verify that each vertex in $H$ is at distance at most two from $h_3$, and that vertices $h_1$, $h_5$, and $h_6$ are the only vertices in $H$ at distance two from $h_3$. Thus there must exist a vertex $z_3$ not in $H$ at distance three from $h_3$ that is adjacent to a subset of $\{h_1, h_5, h_6\}$, and no other vertices of $H$. Further, vertex $z_3$ is not adjacent to $z_v$, otherwise the distance from $z_3$ to $h_3$ is not three. But now $\{z_v, z_3, h_2, h_4\}$ determines an independent set of order four, a contradiction. Now assume that $v$ is adjacent to $h_4$ and $h_6$. It is straightforward to verify that each vertex in $H$ is at distance at most two from $v$, and that vertices $h_3$ and $h_5$ are the only vertices in $H$ at distance two from vertex $v$. Thus, there must exist a vertex $z_v$ not in $H$ at distance three from $v$ that is adjacent to both $h_3$ and $h_5$, and no other vertices of $H$. But now $\{z_v, h_1, h_4, h_6\}$ determines an independent set of order four, a contradiction.

*Case 1b:* Assume that $v$ is not adjacent to $h_2$. If $v$ has exactly three neighbors in $H$, then it is easily verified that its neighborhood restricted to $H$ is a subset of four consecutive vertices, unless a neighbor of $v$ is $h_5$. If the third neighbor is then either $h_4$ or $h_6$, the neighborhood of $v$ restricted to $H$ is a subset of four consecutive vertices. But if the neighbors of $v$ restricted to $H$ are precisely $\{h_1, h_3, h_5\}$, then $\{v, h_2, h_4, h_6\}$ determines an independent set of order four, a contradiction. Now let us assume that $v$ has exactly four neighbors in $H$. In this case there are four possibilities: vertex $v$ is adjacent to each vertex of $\{h_1, h_3, h_4, h_5\}$, $\{h_1, h_3, h_4, h_6\}$, $\{h_1, h_3, h_5, h_6\}$, or $\{h_1, h_4, h_5, h_6\}$. The last set of vertices are consecutive, so we only consider the first three possibilities. In each of these cases, one can verify that each vertex in $H$ is at distance at most two from $v$, and that the two vertices in $H$ not adjacent to $v$ are the only vertices in $H$ at distance two from $v$. Thus there must exist a vertex $z_v$ not in $H$ at distance three from $v$ that is adjacent to the two vertices of $H$ not adjacent to $v$ ($h_2$ and $h_6$, $h_2$ and $h_5$, or $h_2$ and $h_4$, respectively) and no other vertices of $H$. Finally, in each of these cases we will demonstrate an independent set of order four, which contradicts $\alpha = r = 3$. The independent sets are $\{z_v, h_1, h_3, h_5\}$, $\{z_v, h_1, h_4, h_6\}$, or $\{z_v, h_1, h_3, h_6\}$, respectively.

*Case 2:* Suppose that $r \geq 4$. Let $k$ be the smallest integer such that $v$ is adjacent to $h_k$. If $k \geq 4$, then since $\delta(v) \leq 3$, the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. We now consider the three remaining subcases separately.

*Case 2a:* Assume that $k = 3$. If $v$ is not adjacent to $h_{2r}$, then since $\delta(v) \leq 3$, the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. So we will assume that $v$ is adjacent to $h_{2r}$. We will show that $v$ must have at least one other neighbor in $H$, otherwise our assumption $\alpha = r$ will be violated. Suppose $v$ is adjacent to precisely $h_3$ and $h_{2r}$ of $H$. Each vertex in $H$ is at distance at most $r-1$ from vertex $h_3$, and $h_{r+2}$ and $h_{r+3}$ are the only vertices in $H$ at distance $r-1$ from vertex $h_3$. Thus, there must exist a vertex $z_3$ not in $H$ at distance $r$ from $h_3$ that is adjacent to $h_{r+2}$ and $h_{r+3}$, and no other vertices of $H$. If $r$ is even, then $\{h_1, h_3, ..., h_{r+1}, h_{r+4}, h_{r+6}, ..., h_{2r}, z_3\}$ determines an independent set of order $r+1$. Hence we can assume $r > 4$ and $r$ is odd. Then each vertex in $H$ is at distance at most $r-1$ from $v$, and $h_{r+1}$ and $h_{r+2}$ are the only vertices in $H$ at distance $r-1$ from $v$. Thus, there must exist a vertex $z_v$ not in $H$ at distance $r$ from $v$ that is adjacent to $h_{r+1}$ and $h_{r+2}$, and no other vertices of $H$. Then $\{h_1, h_3, ..., h_r, h_{r+3}, h_{r+5}, ..., h_{2r}, z_v\}$ determines an independent set of order $r+1$. Thus, if $k=3$, then $v$ must have another neighbor in $H$ in addition to $h_3$ and $h_{2r}$, otherwise $\alpha \neq r$.

Since $k=3$, $v$ is adjacent to $h_3$ and $h_{2r}$, and $\delta(v) \leq 3$, any other neighbor of $v$ in $H$, say $h_j$, must satisfy both $j-3 \leq 3$ and $2r-j \leq 3$, which imply that $2r-3 \leq j \leq 6$. So if $r \geq 5$, then $v$ could only be adjacent to $h_3$ and $h_{2r}$ of $H$, which we have shown is impossible when $\alpha = r$. Thus, we can assume that $r=4$, and note that any neighbor of $v$ in $H$ other than $h_3$ and $h_{2r}$ has index either 5 or 6. But $v$ cannot be adjacent to either $h_5$ or $h_6$, otherwise the eccentricity of $v$ would be less than four. Thus, when $k=3$, the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices.

*Case 2b:* Assume that $k=2$. If $v$ is not adjacent to $h_{2r}$ nor to $h_{2r-1}$, then since $\delta(v) \leq 3$, the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. So we will assume that $v$ is adjacent to at least one of $h_{2r-1}$ and $h_{2r}$. Moreover, we observe that if $v$ is only adjacent to $h_2$, $h_{2r-1}$, and $h_{2r}$ in $H$, then the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. Thus, we can assume that $v$ has at least one additional neighbor in $H$.

Suppose that $v$ is adjacent to $h_{2r}$. Now since $\delta(v) \leq 3$ and $v$ is adjacent to $h_2$ and $h_{2r}$, it follows that $v$ can only be adjacent to $h_3$ or $h_{2r-1}$ when $r \geq 5$; or $h_3$, $h_5$, or $h_7$ when $r=4$. If $v$ is adjacent to only $h_2$, $h_{2r}$, $h_3$, or only to $h_2$, $h_{2r}$, $h_{2r-1}$ in $H$, then it follows that the neighborhood of $v$ restricted to $H$ is a subset of four consecutive vertices. If $v$ is adjacent to $h_2$, $h_{2r}$ and to both $h_3$ and $h_{2r-1}$, then each vertex in $H$ is at distance at most $r-1$ from $v$, and $h_{r+1}$ is the only vertex that is possibly at distance exactly $r-1$ from $v$. But this implies that the eccentricity of $v$ is less than $r$. Lastly, when $r=4$ and $v$ is adjacent to $h_5$, each vertex in $H$ is at distance at most two from $v$, which implies that the eccentricity of $v$ is less than four.

Next, suppose that $v$ is not adjacent to $h_{2r}$. We previously noted that in this case $v$ must be adjacent to $h_{2r-1}$. Now since $\delta(v) \leq 3$ and $v$ is adjacent to $h_2$ and $h_{2r-1}$, if $r \geq 5$, then $v$ can have no other neighbors in $H$, in which case the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. If $r=4$, then since $\delta(v) \leq 3$ and $v$ is adjacent to $h_2$ and $h_{2r-1}$, $v$ can also be adjacent to either $h_4$ or $h_5$. But if $r=4$ and $v$ is adjacent to either $h_4$ or $h_5$, then each vertex in $H$ is at distance at most two from $v$, which implies that the eccentricity of $v$ is less than four.

*Case 2c:* Assume that $k=1$. If $v$ is not adjacent to any of $h_{2r-2}$, $h_{2r-1}$, or $h_{2r}$, then since $\delta(v) \leq 3$, the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. So we will assume that $v$ is adjacent to at least one of $h_{2r-2}$, $h_{2r-1}$, or $h_{2r}$. Let $k'$ be the smallest integer among $2r-2$, $2r-1$, and $2r$ such that $v$ is adjacent to $h_{k'}$.

Suppose that $k' = 2r-2$. In this case, since $\delta(v) \leq 3$ and $v$ is adjacent to both $h_1$ and $h_{2r-2}$, if $r \geq 5$, then $v$ can have no other neighbors in $H$, in which case the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. If $r=4$, then since $\delta(v) \leq 3$ and $v$ is adjacent to $h_1$ and $h_{2r-2}$, $v$ can be adjacent to either $h_3$ or $h_4$. But if $r=4$ and $v$ is adjacent to either of $h_3$ or $h_4$, then each vertex in $H$ is at distance at most three from $v$, and $h_8$ is the only vertex that is possibly at distance exactly three from $v$. But this implies that the eccentricity of $v$ is less than four.

Suppose that $k' = 2r - 1$. In this case, since $\delta(v) \leq 3$ and $v$ is adjacent to $h_1$ and $h_{2r-1}$, if $r \geq 5$, then $v$ can have at most one other neighbor in $H$, namely $h_2$, in which case the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. If $r = 4$, then since $\delta(v) \leq 3$ and $v$ is adjacent to $h_1$ and $h_{2r-2}$, $v$ can be adjacent to $h_2$ or $h_4$. If $r = 4$ and $v$ is not adjacent to $h_4$, then the neighborhood of $v$ restricted to $H$ is clearly a subset of four consecutive vertices. If $r = 4$ and $v$ is adjacent to $h_4$, then each vertex in $H$ is at distance at most two from $v$, which implies that the eccentricity of $v$ is less than four.

Finally, suppose that $k' = 2r$. In this case, since $\delta(v) \leq 3$ and $v$ is adjacent to $h_1$ and $h_{2r}$ (but not to $h_{2r-1}$ and $h_{2r-2}$), $v$ can have at most two other neighbors in $H$, namely $h_2$ and $h_3$. In this case (which also completes the first claim in the statement of the lemma), the neighborhood of $v$ restricted to $H$ is a subset of four consecutive vertices.

Before proving that if $r \geq 4$, the neighbors of $v$ in $H$ cannot be a subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, let us note that once this is proven, the fact that the neighbors of $v$ in $H$ cannot be a subset of $\{h_1, h_2, h_3, h_{2r}\}$ will follow by a symmetric argument. To prove that the neighbors of $v$ in $H$ cannot be subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, let us suppose otherwise. Since $\delta(v) = 3$, and we have assumed that the set of neighbors of $v$ in $H$ is a subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, it follows that vertex $v$ must be adjacent to vertices $h_1$ and $h_{2r-2}$. Since $r \geq 4$, it is easily verified that $h_{r-2}$ and $h_{r-1}$ are the only vertices in $H$ possibly at distance $r - 1$ from $h_{2r-2}$. Thus, there must exist a vertex $z$ not in $H$ at distance $r$ from $h_{2r-2}$ that is adjacent to both $h_{r-2}$ and $h_{r-1}$, and no other vertices in $H$. If $r$ is even, then the set $\{h_1, h_3, ..., h_{r-3}, h_r, h_{r+2}, ..., h_{2r}, z\}$ determines an independent set of order $r + 1$, which contradicts $\alpha = r$. So suppose that $r$ is odd. Then $r \geq 5$, and we consider the eccentricity of vertex $v$. In this case, since $r \geq 5$, it is easily verified that $h_{r-1}$ and $h_r$ are the only vertices in $H$ possibly at distance $r - 1$ from $v$. Thus, there must exist a vertex $z$ not in $H$ at distance $r$ from $v$ that is adjacent to both $h_{r-1}$ and $h_r$, and no other vertices in $H$. In this case, the set $\{h_1, h_3, ..., h_{r-2}, h_{r+1}, h_{r+3}, ..., h_{2r}, z\}$ determines a independent set of order $r + 1$, again a contradiction to $\alpha = r$.

Next we show that if $r \geq 4$, then the neighbors of $v$ in $H$ cannot be a subset of $\{h_1, h_2, h_{2r-1}, h_{2r}\}$. Let us suppose otherwise. Since $\delta(v) = 3$, and we have assumed that the set of neighbors of $v$ in $H$ is a subset of $\{h_1, h_2, h_{2r-1}, h_{2r}\}$, it follows that $v$ must be adjacent to $h_2$ and $h_{2r-1}$. Since $r \geq 4$, it is easily verified that $h_r$ and $h_{r+1}$ are the only vertices in $H$ possibly at distance $r - 1$ from $v$. Thus, there must exist a vertex $z$ not in $H$ at distance $r$ from $v$ that is adjacent to both $h_r$ and $h_{r+1}$, and no other vertices in $H$. If $r$ is even, the set $\{h_1, h_3, ..., h_{r-1}, h_{r+2}, h_{r+4}, ..., h_{2r}, z\}$ determines an independent set of order $r + 1$, which contradicts $\alpha = r$. Thus, $r$ must be odd. In this case, $r \geq 5$, and since $h_{r-1}$ and $h_r$ are the only vertices in $H$ possibly at distance $r - 1$ from $h_{2r-1}$, there exists a vertex $z$ not in $H$ adjacent to both $h_{r-1}$ and $h_r$, and no other vertices in $H$. In this case, the set $\{h_1, h_3, ..., h_{r-2}, h_{r+1}, h_{r+3}, ..., h_{2r}, z\}$ determines an independent set of order $r + 1$, again a contradiction. $\square$

**Lemma 7.** *Let $G$ be a graph with $\alpha = r = 3$. Suppose $H = P(6)$. Let $U \subset V(G) - V(H)$ be a collection of vertices such that for every $u \in U$, $\delta(u) = 3$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_5, h_6\}$. Then $U$ must induce a clique in $G$, and each vertex $u \in U$ must be adjacent to each of the vertices $\{h_1, h_2, h_5, h_6\}$. Moreover, if there exists a vertex $v \in V(G) - V(H)$ adjacent to $h_6$ such that the neighbors of $v$ in $H$ are a subset of $\{h_1, h_2, h_6\}$, then $v$ is adjacent to each vertex of $U$.*

*Proof.* First, we prove that every vertex $u \in U$ must be adjacent to each of the vertices $\{h_1, h_2, h_5, h_6\}$. For any $u \in U$, since $\delta(u) = 3$ and the neighbors of $u$ that are in $H$ determine a subset of $\{h_1, h_2, h_5, h_6\}$, $u$ must be adjacent to $h_2$ and $h_5$. Further, $u$ must be adjacent to at least one of $h_1$ or $h_6$, otherwise the independent set $\{h_1, h_3, h_6, u\}$ contradicts $\alpha = r = 3$. Now we proceed by contradiction, and without loss of generality, suppose that $u$ is not adjacent to $h_6$. Then let us consider the eccentricity of $h_5$. It is easily verified that there must be a vertex $z$ at distance three from vertex $h_5$ that is adjacent to only a subset of $h_1$, $h_2$, and $h_3$ in $H$. Note that $z$ cannot be adjacent to vertex $u$, otherwise $z$ is not at distance three from $h_5$. But then there exists an independent set of order four, namely $\{u, h_4, h_6, z\}$, which contradicts $\alpha = r = 3$. Thus, every vertex $u \in U$ must be adjacent to each of the vertices $h_1$, $h_2$, $h_5$, and $h_6$.

Next, to prove that $U$ must induce a clique in $G$, let us suppose otherwise. Let $u_1$ and $u_2$ be nonadjacent vertices in $U$. It is easily verified that there must exist a vertex $z$ at distance three from $h_5$ that is adjacent to only a subset of $h_1$, $h_2$, and $h_3$ in $H$. Note that $z$ cannot be adjacent to any of $u_1$, $u_2$, or $h_4$, otherwise it is not at distance three from $h_5$. In this case, $\{u_1, u_2, h_4, z\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$. Thus the vertices of $U$ must induce a clique in $G$.

To prove our last claim, suppose that there exists a vertex $v \in V(G) - V(H)$ adjacent to $h_6$ such that the neighbors of $v$ in $H$ are a subset of $\{h_1, h_2, h_6\}$, but that $v$ is not adjacent to some vertex $u \in U$. We have proven that $u$ must be adjacent to each of the vertices $\{h_1, h_2, h_5, h_6\}$. By assumption, $v$ is adjacent to $h_6$ and if its only other neighbor in $H$ is $h_2$, then the set $\{h_1, h_3, h_5, v\}$ would contradict $\alpha = r = 3$. Thus, $v$ must be adjacent to $h_1$. It is straightforward to verify that each vertex in $H$ is at distance at most two from vertex $u$, and that vertices $h_3$ and $h_4$ are the only vertices in $H$ at distance two from vertex $u$. Thus there must exist a vertex $z_u$ not in $H$ at distance three from $u$ that is adjacent to both $h_3$ and $h_4$, and no other vertices of $H$. Similarly, there must exist a vertex $z_5$ not in $H$ at distance three from $h_5$ that is adjacent to some subset of $h_1$, $h_2$, and $h_3$, and no other vertices of $H$. Vertex $z_5$ must be adjacent to $h_2$, otherwise the set $\{z_5, h_2, h_4, h_6\}$ would contradict $\alpha = r = 3$. Vertex $z_5$ must also be adjacent to $h_1$, otherwise the set $\{z_5, h_1, h_4, h_6\}$ would likewise contradict $\alpha = r = 3$. Moreover, since $u$ is adjacent to $h_5$, $z_5$ is not adjacent to $u$. By a symmetric argument, there must exist a vertex $z_2$ not in $H$ at distance three from $h_2$ that is adjacent to some subset of $h_4$, $h_5$, and $h_6$, and no other vertices of $H$. Vertex $z_2$ must be adjacent to $h_5$, otherwise the set $\{z_2, h_1, h_3, h_5\}$ would contradict $\alpha = r = 3$. Vertex $z_2$ must also be adjacent to $h_6$, otherwise the set $\{z_2, h_1, h_3, h_6\}$ would likewise contradict $\alpha = r = 3$. Moreover, since $u$ is adjacent to $h_2$, $z_2$ is not adjacent to $u$. In order that $\{u, v, h_4, z_5\}$ not determine an independent set of order four, the only possibility is that $v$ and $z_5$ must be adjacent.

Similarly, in order that $\{u,\, v,\, h_3, z_2\}$ not determine an independent set of order four, the only possibility is that $v$ and $z_2$ must be adjacent.

By assumption, the neighbors of $v$ in $H$ are a subset of $\{h_1, h_2, h_6\}$, but since we have shown that $v$ must be adjacent to $z_2$, it follows that $v$ cannot be adjacent to $h_2$, otherwise $z_2$ would not be at distance three from $h_2$. In order that $\{h_2, h_5, v, z_u\}$ not determine an independent set of order four, the only possibility is that $v$ and $z_u$ are adjacent. In this case, let us consider the eccentricity of $v$. It is at distance at most two from each vertex of $H \cup \{u, z_u, z_2, z_5\}$, and distance two from all vertices in $H$ except $h_1$ and $h_6$. Thus, there exists a vertex $z_v$ adjacent to a subset of vertices of $H$ that does not include the vertices $h_1$ and $h_6$. In this case, in order that $\{h_1, h_6, z_v,\, z_u\}$ not determine an independent set of order four, the only possibility is that $z_v$ and $z_u$ are adjacent. But this contradicts that $z_v$ is at distance three from $v$, since we had earlier deduced that $v$ and $z_u$ are adjacent. $\square$

**Lemma 8.** *Let $G$ be a graph with $\alpha = r = 3$. Suppose $H = P(6)$. Let $U \subset V(G) - V(H)$ be a collection of vertices such that for every $u \in U$, $\delta(u) = 3$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_4, h_5, h_6\}$. Then $U$ must induce a clique in $G$, and each vertex $u \in U$ must be adjacent to exactly the set $\{h_1, h_4, h_5\}$ in $H$. Moreover, there exists a vertex $v \in V(G) - V(H)$ adjacent to exactly the set $\{h_1, h_5, h_6\}$ in $H$, and each such vertex $v$ is adjacent to each vertex of $U$.*

*Proof.* For any $u \in U$, since $\delta(u) = 3$ and the neighbors of $u$ that are in $H$ determine a subset of $\{h_1, h_4, h_5, h_6\}$, $u$ must be adjacent to $h_1$ and $h_4$. First, let us prove that $u$ cannot be adjacent to $h_6$. Suppose otherwise. In this case, $u$ is at distance at most two from each vertex in $H$, and the only vertices in $H$ at distance two from vertex $u$ are $h_2$, $h_3$, and possibly $h_5$. Thus there exists a vertex $z$ at distance three from vertex $u$ that is adjacent to some subset of the vertices $h_2$, $h_3$, and $h_5$, and to no other vertices in $H$. But now $\{h_1, h_4, h_6, z\}$ determines an independent set of order four, a contradiction to $\alpha = r = 3$. Thus the neighbors of $u$ that are in $H$ determine a subset of $\{h_1, h_4, h_5\}$. Next, suppose that $u$ is not adjacent to $h_5$. Then we consider the eccentricity of $h_4$. Since it is at distance at most two from each vertex in $H$, and at distance exactly two from vertices $h_1$, $h_2$, and $h_6$, there exists a vertex $z$ at distance three from $h_4$ that is adjacent to some subset of $\{h_1, h_2, h_6\}$, and to no other vertices in $H$. Moreover, $z$ is not adjacent to $u$, otherwise $z$ is not at distance three from $h_4$. But now $\{h_3, h_5, u, z\}$ is an independent set that contradicts $\alpha = r = 3$. Thus, $u$ must be adjacent to exactly the set $\{h_1, h_4, h_5\}$ in $H$.

Next, to prove that $U$ must induce a clique in $G$, let us suppose otherwise. Let $u_1$ and $u_2$ be nonadjacent vertices in $U$. It is easily verified that there must exist a vertex $z$ at distance three from $h_4$ that is adjacent to only a subset of $\{h_1, h_2, h_6\}$ in $H$. Moreover, $z$ cannot be adjacent to either $u_1$ or $u_2$, otherwise $z$ is not at distance three from $h_4$. In this case, $\{u_1, u_2, h_3, z\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$. Thus the vertices of $U$ must induce a clique in $G$.

To prove our last claim, note that we have proven that a vertex $u \in U$ must be adjacent to exactly the set $\{h_1, h_4, h_5\}$ in $H$. It is straightforward to verify that each vertex in $H$ is at distance at most two from $h_4$, and that vertices $\{h_1, h_2, h_6\}$ are the only vertices in $H$ at distance two from vertex $h_4$. Thus there must exist a vertex $z_4$ not in $H$ at distance three from $h_4$ that is adjacent to some subset of $\{h_1, h_2, h_6\}$, and no other vertices of $H$.

Moreover, since $u$ is adjacent to $h_4$, $z_4$ is not adjacent to vertex $u$, otherwise the distance from $h_4$ to $z_4$ is not three as assumed. Vertex $z_4$ must be adjacent to both of $h_1$ and $h_6$, otherwise one of $\{u, z_4, h_3, h_6\}$ or $\{h_1, h_3, h_5, z_4\}$ contradicts $\alpha = r = 3$. Similarly, it is easy to verify that each vertex in $H$ is at distance at most two from vertex $u$, and that vertices $\{h_2, h_3, h_6\}$ are the only vertices in $H$ at distance two from vertex $u$. Thus, there must exist a vertex $z_u$ not in $H$ at distance three from $u$ that is adjacent to some subset of $\{h_2, h_3, h_6\}$, and no other vertices of $H$. Vertex $z_u$ must be adjacent to both of $h_3$ and $h_6$, otherwise one of $\{h_1, h_3, h_5, z_u\}$ or $\{h_1, h_4, h_6, z_u\}$ contradicts $\alpha = r = 3$. We prove the following claim.

*Claim:* $z_u$ is adjacent to $h_2$. By way of contradiction, suppose that $z_u$ is not adjacent to $h_2$. One can check that each vertex in $H$ is at distance at most two from vertex $h_3$, and the vertices $\{h_1, h_5, h_6\}$ are the only vertices in $H$ at distance two from vertex $h_3$. Thus, there must exist a vertex $z_3$ not in $H$ at distance three from $h_3$ that is adjacent to some subset of $\{h_1, h_5, h_6\}$, and no other vertices of $H$. Moreover, since $z_u$ is adjacent to $h_3$, $z_3$ is not adjacent to $z_u$, otherwise $z_3$ and $h_3$ are not at distance three as assumed. But now $\{h_2, h_4, z_u, z_3\}$ contradicts $\alpha = r = 3$. Thus, $z_u$ is adjacent to $h_2$.

In this case, each vertex in $H$ is at distance at most two from $h_6$, and vertices $\{h_1, h_2, h_3, h_4\}$ are the only vertices in $H$ at distance two from vertex $h_6$. Thus, there must exist a vertex $z_6$ not in $H$ at distance three from $h_6$ that is adjacent to some subset of $\{h_1, h_2, h_3, h_4\}$, and no other vertices of $H$. Since $z_4$ and $z_u$ are adjacent to $h_6$, $z_6$ is not adjacent to either $z_4$ or $z_u$, otherwise the distance from $h_6$ to $z_6$ is not three as assumed. So we see that $z_4$ must be adjacent to $z_u$, otherwise $\{z_4, z_6, z_u, h_5\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$.

Since $z_6$ must be adjacent to two vertices in $H$, each vertex of $H \cup \{u, z_u, z_4, z_6\}$ is at distance at most two from vertex $h_3$, and vertices $h_1$, $h_5$, and $h_6$ are the only vertices in $H$ at distance two from vertex $h_3$. Thus, there must exist a vertex $z_3$ not in $H$ at distance three from $h_3$ that is adjacent to some subset of $\{h_1, h_5, h_6\}$, and no other vertices of $H$. At this point, we note that the focus of the remainder of our proof is to demonstrate that $z_3$ is the vertex $v$ that exists as claimed in the statement of the lemma. Vertex $z_3$ must be adjacent to $h_6$, otherwise $\{h_2, h_4, h_6, z_3\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$. Since $z_u$ is adjacent to $h_3$, vertices $z_u$ and $z_3$ are not adjacent, otherwise the distance from $h_3$ to $z_3$ is not three as assumed. Thus, we can now argue that $z_3$ must be adjacent to $h_1$, otherwise $\{h_1, h_4, z_3, z_u\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$. Finally, we see that $z_3$ must indeed be adjacent to $h_5$ as well. Otherwise $\{z_3, z_6, z_u, h_5\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$ (note that $z_3$ cannot be adjacent to $z_6$). Thus, there is a vertex $v \in V(G) - V(H)$ adjacent to exactly the set $\{h_1, h_5, h_6\}$ in $H$.

To complete the proof, let us observe that $z_6$ must be adjacent to $h_1$, otherwise $\{h_1, h_5, z_u, z_6\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$. Now it is easily verified that each vertex in $H$ is at distance at most two from $h_1$, and that vertices $\{h_3, h_4, h_5, h_6\}$ are the only vertices in $H$ at distance two from $h_1$. Thus, there must exist a vertex $z_1$ not in $H$ at distance three from $h_1$ that is adjacent to some subset of $\{h_3, h_4, h_5, h_6\}$, and no other vertices of $H$. Since vertices $u$ and $z_3$ are adjacent to $h_1$, $z_1$ is not adjacent to either $u$ or $z_3$, otherwise the distance from $h_1$ to $z_1$

is not three as assumed. It follows that $z_3$ must be adjacent to $u$, otherwise $\{u, z_1, z_3, h_2\}$ will determine an independent set of order four, which contradicts $\alpha = r = 3$. $\square$

**Lemma 9.** *Let $G$ be a graph with $\alpha = r = 3$. Suppose $H = P(6)$. Let $U \subset V(G) - V(H)$ be a collection of vertices such that for every $u \in U$, $\delta(u) = 3$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_3, h_6\}$. Then $U$ must induce a clique in $G$, and each vertex $u \in U$ must be adjacent to exactly the set $\{h_2, h_3, h_6\}$ in $H$. Moreover, there exists a vertex $v \in V(G) - V(H)$ adjacent to exactly the set $\{h_1, h_2, h_6\}$ in $H$, and each such vertex $v$ is adjacent to each vertex of $U$.*

*Proof.* The proof is symmetric to the proof of Lemma 8. $\square$

**Lemma 10.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Assume $H = P(2r)$. Suppose $U$ is a collection of vertices such that $U \subset V(G) - V(H)$ and $k =$min$\{j \mid u \in U$ and $u$ is adjacent to $h_j\}$. Moreover, suppose for every $u \in U$ that $u$ is adjacent to $h_k$, and $\delta(v) = 3$ for some $v \in U$. Then:*

*1) If $2 \leq k \leq 2r-4$, then there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to both $h_1$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these two vertices in $H$, and $z$ is not adjacent to any vertex $u \in U$.*

*2) If $k = 1$ and for every vertex $u \in U$, $u$ is adjacent to $h_4$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_3, h_4\}$, then there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_1$ and at least one of $h_2$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is not adjacent to any vertex $u \in U$.*

*3) If $k = 2r - 3$ and for every vertex $u \in U$, the neighbors of $u$ in $H$ are a subset of $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$, then there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_{2r}$ and at least one of $h_1$ or $h_{2r-1}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is not adjacent to any vertex $u \in U$.*

*Proof.* By Lemma 2, we know that $v$ is adjacent to at least two vertices in $H$. First, we make the following simple but useful observations.

*) If $k \geq 4$, then since $\delta(v) = 3$, $v$ is adjacent to $h_{k+3}$.

**) If $k \leq 3$, then since $\delta(v) = 3$, $v$ is adjacent to $h_{k+3}$ or $h_{2r-(3-k)}$.

*Proof of 1).* Recall we are assuming that $2 \leq k \leq 2r - 4$. First, suppose $r = 3$. Then $k = 2$, and by **, $v$ is adjacent to $h_2$ and also $h_5$. We begin by showing that $v$ is not adjacent to $h_6$, so suppose otherwise. In this case, it is straightforward to verify that each vertex in $H$ is at distance at most two from $h_2$, and that $h_4$, $h_5$, and $h_6$ are the only vertices in $H$ possibly at distance two from $h_2$. Thus, there must exist a vertex $z_2$ not in $H$ at distance three from $v$ that is adjacent to a subset of $h_4$, $h_5$, and $h_6$, and no other vertices of $H$. Clearly, $z_2$ is not adjacent to $v$, otherwise $h_2$ and $z_2$ are not at distance three, as assumed. But now, the set $\{h_1, h_3, v, z_2\}$ contradicts $\alpha = r = 3$. Next, observe that $v$ must adjacent to both $h_3$ and $h_4$, otherwise an independent set of order four is easily found, namely $\{h_1, h_3, h_6, v\}$ or $\{h_1, h_4, h_6, v\}$. Now, it is straightforward to verify that each vertex in $H$ is at distance at most two from $v$, and that $h_1$ and $h_6$ are the only vertices in $H$ at distance two from $v$. Thus, there must exist a vertex $z$ not in $H$ at distance three from $v$ that is adjacent to both $h_1$ and $h_6$, and no other vertices of $H$. Clearly, $z$ is not adjacent to $v$, otherwise they are not at distance three, as assumed. If there is some $u \in U$ that is adjacent to $z$, then by assumption $u$ is also adjacent to $h_2$

but not to $v$ (otherwise $z$ and $v$ are not at distance three). Vertex $u$ must be adjacent to $h_6$, otherwise the set $\{u, v, h_1, h_6\}$ contradicts $\alpha = r = 3$. Now, it is easily verified that there must be a vertex $z_2$ not in $H$ at distance three from $h_2$. Since $z_2$ is not adjacent to $v$, $u$, and $h_1$, the set $\{u, v, h_1, z_2\}$ contradicts $\alpha = r = 3$. Thus, 10.1) is true when $r = 3$.

Next suppose that $r \geq 4$, and let us consider five cases.

*Case 1a:* Suppose that $k = 2$. Then by **, $v$ may be adjacent to $h_5$ or $h_{2r-1}$. By Lemma 6, we see that $v$ cannot be adjacent to both $h_5$ and $h_{2r-1}$, otherwise the neighborhood of $v$ restricted to $H$ is a not subset of four consecutive vertices.

First, we will show that $v$ cannot be adjacent to $h_{2r-1}$. So suppose otherwise. In this case, by Lemma 6, we see that the only other possible neighbor of $v$ is $h_{2r}$, otherwise the neighborhood of $v$ in $H$ is not a subset of four consecutive vertices. Observe that $v$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_r$ and $h_{r+1}$ are the only vertices possibly at distance $r-1$ from $v$ in $H$. Thus, there is a vertex $z_v$ at distance $r$ from $v$ that is adjacent to $h_r$ and $h_{r+1}$, and no other vertices of $H$. Now, if $r$ is even, then the set $\{h_1, h_3, ..., h_{r-1}, h_{r+2}, h_{r+4}, ..., h_{2r}, z_v\}$ determines an independent set of order $r+1$, which contradicts $\alpha = r$. If $r$ is odd, then observe $h_2$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_{r+1}$ and $h_{r+2}$ are the only vertices possibly at distance $r-1$ from $v$ in $H$. Thus, there is a vertex $z_2$ at distance $r$ from $h_2$ that is adjacent to $h_{r+1}$ and $h_{r+2}$, and no other vertices of $H$. In this case, the set $\{h_1, h_3, ..., h_r, h_{r+3}, ..., h_{2r}, z_3\}$ determines an independent set of order $r+1$, which contradicts $\alpha = r$.

Next, suppose that $v$ is adjacent to $h_5$. By Lemma 6, we see that the only other neighbors of $v$ are $h_3$ and $h_4$, otherwise the neighborhood of $v$ in $H$ is not a subset of four consecutive vertices. Next, observe that $v$ must be adjacent to both $h_3$ and $h_4$, otherwise an independent set of order $r+1$ is easily found, namely $\{h_1, h_3, h_6, ..., h_{2r}, v\}$ or $\{h_1, h_4, h_6, ..., h_{2r}, v\}$. Now, it is easily verified that $h_{r+1}$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_1$ and $h_{2r}$ are the only vertices in $H$ possibly at distance $r-1$ from $h_{r+1}$. Thus, there is a vertex $z = z_{r+1}$ at distance $r$ from $h_{r+1}$ that is adjacent to $h_1$ and $h_{2r}$, and no other vertices of $H$. Clearly, $z_{r+1}$ is not adjacent to $v$, otherwise $h_{r+1}$ and $z_{r+1}$ are not at distance $r$, as assumed. If there is some $u \in U$ that is adjacent to $z_{r+1}$, then by assumption $u$ is also adjacent to $h_2$, but not to any of $h_4, h_5, ..., h_{2r-2}$ (otherwise, $h_{r+1}$ and $z_{r+1}$ are not at distance $r$). Now, in order that the set $\{h_1, h_4, h_6, ..., h_{2r}, u\}$ not be an independent set of order $r+1$, $u$ must be adjacent to $h_{2r}$. Vertex $u$ may be adjacent to $h_3$ or $h_{2r-1}$, but not to both, since by Lemma 6 the neighborhood of $u$ restricted to $H$ must be a subset of four consecutive vertices. If $u$ is adjacent to $h_3$, then it is easily verified that $u$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_{r+1}$ and $h_{r+2}$ are the only vertices in $H$ possibly at distance $r-1$ from $u$. Thus, there is a vertex $z_u$ at distance $r$ from $u$ that is adjacent to $h_{r+1}$ and $h_{r+2}$, and no other vertices of $H$. Similarly, it is easily verified that $h_3$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_{r+2}$ and $h_{r+3}$ are the only vertices in $H$ possibly at distance $r-1$ from $h_3$. Thus, there is a vertex $z_3$ at distance $r$ from $h_3$ that is adjacent to $h_{r+2}$ and $h_{r+3}$, and no other vertices of $H$. Now, if $r$ is odd, then the set $\{h_1, h_3, ..., h_r, h_{r+3}, h_{r+5}, ..., h_{2r}, z_u\}$ determines an independent set of order $r+1$, which contradicts $\alpha = r$. And, if $r$ is even, then the set $\{h_1, h_3, ..., h_{r+1}, h_{r+4}, h_{r+6}, ..., h_{2r}, z_3\}$ determines an independent set of order $r+1$, which contradicts $\alpha = r$. Thus if $k = 2$, $v$ is adjacent to $h_5$, and $r \geq 4$, then 10.1) holds.

*Case 1b:* Suppose that $k = 3$. Then by **, $v$ may be adjacent to $h_6$ or $h_{2r}$. By Lemma 6, we see that $v$ cannot be adjacent to both $h_6$ and $h_{2r}$, otherwise the neighborhood of $v$ restricted to $H$ is a not subset of four consecutive vertices.

First, we will show that $v$ is not adjacent to $h_{2r}$. So suppose otherwise. In this case, by Lemma 6 we see that $v$ can have no other neighbors in $H$, otherwise the neighborhood of $v$ in $H$ is not a subset of four consecutive vertices. Next, it easily verified that $v$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_{r+1}$ and $h_{r+2}$ are the only vertices possibly at distance $r-1$ from vertex $v$ in $H$. Thus, there is a vertex $z_v$ at distance $r-1$ from vertex $v$ that is adjacent to $h_{r+1}$ and $h_{r+2}$, and no other vertices of $H$. If $r$ is even, then the set $\{h_2, h_4, ..., h_r, h_{r+3}, ..., h_{2r-1}, z_v, v\}$ determines an independent set of order $r+1$, which contradicts $\alpha = r$. Now, observe that $h_3$ is at distance at most $r-1$ from all vertices of $H$, and that $h_{r+2}$ and $h_{r+3}$ are the only vertices possibly at distance $r-1$ from vertex $h_3$ in $H$. Thus, there is a vertex $z_3$ at distance $r-1$ from $h_3$ that is adjacent to $h_{r+2}$ and $h_{r+3}$, and no other vertices of $H$. So, if $r$ is odd, then the set $\{h_2, h_4, ..., h_{r+1}, h_{r+4}, ..., h_{2r-1}, z_3\}$ determines an independent set of order $r+1$, which contradicts $\alpha = r$. Thus, we assume that $v$ is adjacent to $h_6$. By Lemma 6, we see that the only other neighbors of $v$ in $H$ are $h_4$ and $h_5$, otherwise the neighborhood of $v$ in $H$ is not a subset of four consecutive vertices. If $r = 4$, then it is easily verified that $v$ is at distance at most three from all vertices of $H$. Moreover, $h_1$ and $h_{2r}$ are the only vertices possibly at distance three from vertex $u$ in $H$. Thus, there is a vertex $z = z_v$ at distance four from vertex $v$ that is adjacent to $h_1$ and $h_{2r}$, and no other vertices of $H$. If there is some $u \in U$ that is adjacent to $z_v$, then by assumption $u$ is also adjacent to $h_3$, but now $v$ and $z_v$ are at distance three, which contradicts our assumption that they were at distance four. On the other hand, if $r \geq 5$, then it is easily verified that $h_{r+1}$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_1$ and $h_{2r}$ are the only vertices possibly at distance $r-1$ from vertex $h_{r+1}$. Thus, there is a vertex $z = z_{r+1}$ at distance $r$ from $h_{r+1}$ that is adjacent to $h_1$ and $h_{2r}$, and no other vertices of $H$. If there is some $u \in U$ that is adjacent to $z_{r+1}$, then by assumption $u$ is also adjacent to $h_3$, but now $h_{r+1}$ and $z_{r+1}$ are at distance four, which contradicts our assumption that they are at distance $r \geq 5$. Thus if $k = 3$, $v$ is adjacent to $h_6$, and $r \geq 4$, then 10.1) holds.

*Case 1c:* Suppose that $4 \leq k \leq r-2$. Then $r \geq 6$ and $7 \leq k+3 \leq r+1$. By *, $v$ is adjacent to $h_{k+3}$. Now it is easily verified that $h_{r+1}$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_1$ and $h_{2r}$ are the only vertices in $H$ possibly at distance $r-1$ from $h_{r+1}$. Thus, there is a vertex $z = z_{r+1}$ at distance $r$ from $h_{r+1}$ that is adjacent to $h_1$ and $h_{2r}$, and no other vertices of $H$. Clearly, $v$ is not adjacent to $z_{r+1}$, otherwise $h_{r+1}$ and $z_{r+1}$ are not at distance $r$ as assumed. Now assume, by way of contradiction, that $z_{r+1}$ is adjacent to some $u \in U$. By definition of the set $U$, $u$ is adjacent to $h_k$. But now the distance between $z_{r+1}$ and $h_{r+1}$ is less than $r$, which contradicts our assumption that the distance from $z_{r+1}$ to $h_{r+1}$ is $r$.

*Case 1d:* If $k = r-1$, then $k+3 = r+2$ and the distance from $v$ to all vertices of $H$ is at most $r-1$. Moreover, $h_1$and $h_{2r}$ are the only vertices possibly at distance $r-1$ from $v$. Thus, there is a vertex $z = z_v$ at distance $r$ from $v$ that is adjacent to $h_1$ and $h_{2r}$, and clearly $v$ is not adjacent to $z_v$. Now assume, by way of contradiction, that $z_v$ is adjacent to some $u \in U$. By definition of the set $U$, $u$ is adjacent to $h_k$. But now the distance between $v$ and $z_v$ is three, which contradicts our assumption that the distance from $z_v$ to $v$ is $r \geq 4$.

*Case 1e:* Suppose that $r \leq k \leq 2r-4$. Then $r+3 \leq k+3 \leq 2r-1$. In this case, let us consider the eccentricity of $h_r$. Vertex $h_r$ is at distance at most $r-1$ from all vertices of $H$. Moreover, $h_1$ and $h_{2r}$ are the only vertices possibly at distance $r-1$ from $h_r$. Thus, there is a vertex $z = z_r$ at distance $r$ from $h_r$ that is adjacent to $h_1$ and $h_{2r}$, and clearly $v$ is not adjacent to $z_r$. Now assume, by way of contradiction, that $z_r$ is adjacent to some $u \in U$. By definition of the set $U$, $u$ is adjacent to $h_k$. But now the distance between $z_r$ and $h_r$ is less than $r$, which contradicts our assumption that the distance from $z_r$ to $h_r$ is $r$.

*Proof of 2).* Assume that $k = 1$ and for every vertex $u \in U$, $u$ is adjacent to $h_4$ and the neighbors of $u$ in $H$ are a subset of $\{h_1, h_2, h_3, h_4\}$. Because $\delta(v) = 3$, $v$ is adjacent to $h_1$ and $h_4$. Let us consider the eccentricity of $h_{r+1}$. Since $r \geq 3$, all vertices of $H$ are at distance at most $r-1$ from $h_{r+1}$, and $h_1$, $h_2$, and $h_{2r}$ are the only vertices in $H$ possibly at distance $r-1$ from $h_{r+1}$. Thus, there must exist a vertex $z = z_{r+1}$ adjacent to at least two of $h_1$, $h_2$, and $h_{2r}$ in $H$. If $z_{r+1}$ is adjacent to only $h_2$ and $h_{2r}$, then $\{h_1, h_3, h_5, ..., h_{2r-1}, z_{r+1}\}$ determines an independent set of order $r+1$, a contradiction to $\alpha = r$. Thus, $z_{r+1}$ must be adjacent to $h_1$. Since $z_{r+1}$ must have two neighbors in $H$, $z_{r+1}$ must be adjacent to at least one of $h_2$ or $h_{2r}$. Clearly, $z$ is not adjacent to $v$. Now, assume by way of contradiction, that $z_{r+1}$ is adjacent to some $u \in U$. By definition of the set $U$, $u$ is adjacent to $h_4$. But now the distance between $z_{r+1}$ and $h_{r+1}$ is less than $r$, which contradicts our assumption that the distance from $z_{r+1}$ to $h_{r+1}$ is $r$.

*Proof of 3).* Assume that $k = 2r-3$ and for every vertex $u \in U$, the neighbors of $u$ in $H$ are a subset of $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$. Because $\delta(v) = 3$, $v$ is adjacent to $h_{2r-3}$ and $h_{2r}$. Let us consider the eccentricity of $h_r$. Since $r \geq 3$, all vertices of $H$ are at distance at most $r-1$ from $h_r$, and $h_1$, $h_{2r-1}$, and $h_{2r}$ are the only vertices in $H$ possibly at distance $r-1$ from $h_r$. Thus, there must exist a vertex $z = z_r$ adjacent to at least two of $h_1$, $h_{2r-1}$, and $h_{2r}$ in $H$. If $z_r$ is adjacent only to $h_1$ and $h_{2r-1}$ in $H$, then $\{h_2, h_4, h_6, ..., h_{2r}, z_r\}$ determines an independent set of order $r+1$, a contradiction to $\alpha = r$. Thus, $z_r$ must be adjacent to $h_{2r}$. Since $z_r$ must have two neighbors in $H$, $z_r$ must be adjacent to at least one of $h_1$ or $h_{2r-1}$. Clearly, $z_r$ is not adjacent to $v$. Now, assume by way of contradiction, that $z_r$ is adjacent to some $u \in U$. By definition of the set $U$, $u$ is adjacent to $h_{2r-3}$. But now the distance between $z_r$ and $h_r$ is less than $r$, which contradicts our assumption that the distance from $z_r$ to $h_r$ is $r$. □

**Lemma 11.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Moreover, suppose $v$ is a vertex such that $v \in V(G) - V(H)$ and the neighbors of $v$ include neither $h_1$ nor $h_{2r}$. Then $v$ is adjacent to exactly two, exactly three, or exactly four consecutive vertices in $H$.*

*Proof.* Let $k$ be the smallest integer such that $v$ is adjacent to $h_k$. Then clearly $k \geq 2$. By Lemma 5, $\delta(v) \leq 3$. If $\delta(v) = 3$, then by Lemmas 6, 7, and 8, $2 \leq k \leq 2r-4$. By Lemma 10.1), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to both $h_1$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these two vertices in $H$, and $z$ is not adjacent to $v$. In this case, we first prove the following claim.

*Claim.* $v$ is adjacent to four vertices in $H$. By way of contradiction, assume $v$ is not adjacent to at least one of $h_{k+1}$ or $h_{k+2}$. Suppose $k$ is even. Then the set $\{h_1, h_3, ..., h_{k-1}, h_{k+1}, h_{k+4}, ..., h_{2r}\} \cup \{v\}$ (in case $v$ is not adjacent to $h_{k+1}$), or the set $\{h_1, h_3, ..., h_{k-1}, h_{k+2}, h_{k+4}, ..., h_{2r}\} \cup \{v\}$ (in case $v$ is not adjacent to $h_{k+2}$) is an independent set of order $r+1$, a contradiction. On the other hand, if $k$ is odd, then the set $\{h_2, h_4, ..., h_{k-1}, h_{k+1}, h_{k+4}, ..., h_{2r-1}\} \cup \{v, z\}$ (in case $v$ is not adjacent to $h_{k+1}$), or the set $\{h_2, h_4, ..., h_{k-1}, h_{k+2}, h_{k+4}, ..., h_{2r-1}\} \cup \{v, z\}$ (in case $v$ is not adjacent to $h_{k+2}$) is an independent set of order $r+1$, a contradiction. Thus, the claim is correct.

Now, if $v$ has four neighbors in $H$, then since $\delta(v) = 3$, the four neighbors are clearly consecutive. Suppose $v$ has three neighbors in $H$. Then by the claim, $\delta(v) = 2$, otherwise $v$ would be forced to have four neighbors in $H$. Thus, if $v$ has three neighbors in $H$, they must be consecutive. Last, suppose that vertex $v$ has two neighbors in $H$. By the claim, $\delta(v) \leq 2$, otherwise $v$ would be forced to have four neighbors in $H$. If $\delta(v) = 2$, then $v$ is adjacent to $h_k$ and $h_{k+2}$. In this case, we have a contradiction, since when $k$ is even $\{h_1, h_3, ..., h_{k-1}, h_{k+1}, h_{k+3}, ..., h_{2r-1}\} \cup \{v\}$ is an independent set of order $r+1$, and when $k$ is odd $\{h_2, h_4, ..., h_{k-1}, h_{k+1}, h_{k+3}, ..., h_{2r}\} \cup \{v\}$ is an independent set of order $r+1$. Thus, $\delta(v) = 1$, and the two neighbors of $v$ are consecutive. □

**Lemma 12.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Moreover, suppose $v$ is a vertex such that $v \in V(G) - V(H)$ and the neighbors of $v$ include $h_1$. Then either:*

*1) $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$; or*
*2) $v$ is adjacent to exactly $h_1$, $h_2$, $h_3$, and $h_4$ in $H$; or*
*3) $v$ is adjacent to $h_1$, $h_3$, and $h_4$; or*
*4) $r = 3$ and $v$ is adjacent to exactly $h_1$, $h_2$, $h_5$, and $h_6$ in $H$; or*
*5) $r = 3$ and $v$ is adjacent to exactly $h_1$, $h_4$, and $h_5$ in $H$.*

*Proof.* First, suppose that $\delta(v) = 1$. Then $v$ is clearly adjacent to exactly two consecutive vertices in $H$ (namely $h_1$ and $h_{2r}$, or $h_1$ and $h_2$).

Next, suppose that $\delta(v) = 2$ but $v$ is not adjacent to three consecutive vertices in $H$. Let $h_m$ and $h_n$ be the two vertices in $H$ adjacent to $v$ so that $\delta(h_m, h_n) = 2$, and let $c$ be the vertex in $H$ between $h_m$ and $h_n$ (consecutive to both of them) which is not adjacent to $v$. Since $r \geq 3$, by Lemma 6, $v$ cannot be adjacent to any other vertices of $H$ other than $h_m$ and $h_n$. We can form an independent set with $\alpha + 1$ vertices by including $v$ with a maximum independent set in $H$ containing $c$ but not containing either $h_m$ or $h_n$. This is a contradiction. Thus, if $\delta(v) = 2$, $v$ is adjacent to three consecutive vertices in $H$.

Last, suppose that $\delta(v) = 3$. Then $v$ cannot be adjacent to exactly two or exactly three consecutive vertices in $H$. We consider the cases $r \geq 4$ and $r = 3$ separately. If $r \geq 4$, by Lemma 6, we know that the neighbors of $v$ in $H$ must be a subset of $\{h_1, h_2, h_3, h_4\}$. Now, $v$ must be adjacent to $h_4$ (because $\delta(v) = 3$), and it must also be adjacent to $h_1$ by hypothesis. If $v$ is adjacent to each of the vertices $\{h_1, h_2, h_3, h_4\}$, we are done. Therefore, suppose that $v$ is not adjacent to $h_3$. Now, by Lemma 10.2), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_1$ and at least one of $h_2$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is not adjacent to $v$. Consequently, we can form an independent set including $v$, $z$, and $h_3$, together with

$\{h_5, h_7, ..., h_{2r-1}\}$, which has $\alpha + 1$ vertices, a contradiction. Thus, $v$ must be adjacent to $h_1$, $h_3$, and $h_4$, the desired result.

On the other hand, if $r = 3$, then the neighbors of $v$ in $H$ must be a subset of $\{h_1, h_2, h_3, h_4\}$, $\{h_1, h_4, h_5, h_6\}$, or $\{h_1, h_2, h_5, h_6\}$. If these neighbors are a subset of $\{h_1, h_2, h_3, h_4\}$, we can argue as in the preceding case to get the desired result. If they are a subset of $\{h_1, h_4, h_5, h_6\}$, then by Lemma 8, $v$ must be adjacent to exactly $h_1$, $h_4$, and $h_5$ in $H$. Finally, if they are a subset of $\{h_1, h_2, h_5, h_6\}$, then by Lemma 7, $v$ must be adjacent to exactly $h_1$, $h_2$, $h_5$, and $h_6$ in $H$. $\square$

**Lemma 13.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Suppose $H = P(2r)$. Moreover, suppose $v$ is a vertex such that $v \in V(G) - V(H)$ and the neighbors of $v$ include $h_{2r}$. Then either:*

*1) $v$ is adjacent to exactly two or exactly three consecutive vertices in $H$; or*

*2) $v$ is adjacent to exactly $h_{2r-3}$, $h_{2r-2}$, $h_{2r-1}$, and $h_{2r}$ in $H$; or*

*3) $v$ is adjacent to exactly $h_{2r-3}$, $h_{2r-2}$, and $h_{2r}$ in $H$; or*

*4) $r = 3$ and $v$ is adjacent to exactly $h_1$, $h_2$, $h_5$, and $h_6$ in $H$; or*

*5) $r = 3$ and $v$ is adjacent to exactly $h_2$, $h_3$, and $h_6$ in $H$.*

*Proof.* The proof is symmetric to the proof of Lemma 12. $\square$

**Lemma 14.** *Let $G$ be a graph with $r \geq 3$ such that $\alpha = r$. Then there exists a subgraph $H$ of $G$ such that either $H = P(2r)$ or $H = C(2r)$, and if $u, v \in V(G) - V(H)$ is a pair of degenerate vertices with respect to $H$, $u$ is adjacent to $v$.*

*Proof.* If $r \geq 5$ and $G$ contains an induced $C(2r)$ subgraph, let $H$ be this subgraph. Then $G$ and $H$ satisfy Lemma 3. If $r \geq 5$ and $G$ does not contain an induced $C(2r)$ subgraph, let $H$ be the induced $P(2r)$ subgraph implied by Lemma 1. If $3 \leq r \leq 4$ and $G$ contains an induced $C(2r)$ subgraph that satisfies Lemma 4, let $H$ be this subgraph. If $3 \leq r \leq 4$ and $G$ does not contain an induced $C(2r)$ subgraph that satisfies Lemma 4, let $H$ be the induced $P(2r)$ subgraph implied by Lemma 4.

If $H = C(2r)$, then $G$ and $H$ satisfy either Lemma 3 or Lemma 4. Thus, the union of the neighbors of $u$ and $v$ in $H$ is three or less consecutive vertices. Let $X$ be this union. Then the subgraph induced by $V(H) - X$ has an independent set of size $r - 1$. Thus if $u$ and $v$ are not adjacent, $G$ has an independent set of size $r + 1$, a contradiction.

Therefore, assume $H = P(2r)$. As in the preceding case, if the union $X$ of the neighbors of $u$ and $v$ in $H$ is three or less consecutive vertices and $u$ and $v$ are not adjacent, then $G$ has an independent set of size $r + 1$, a contradiction. On the other hand, suppose $X$ is not three or less consecutive vertices. We consider four cases, which correspond to the four remaining clauses b) through e) in the definition of degenerate vertices.

*Case 1:* $X$ is four or less consecutive vertices including neither $h_1$ nor $h_{2r}$, and $k = k'$. By our suppositions and Lemma 5, we have that $1 \leq \delta(u), \delta(v) \leq 3$, and either $\delta(u) = 3$ or $\delta(v) = 3$. Otherwise, $X$ is three or less consecutive vertices. Assume $\delta(u) = 3$. In addition, by the definition of $k$, we have $k < 2r - 3$. Then by Lemma 10.1), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to both $h_1$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is adjacent to neither $u$ nor $v$. Since $\delta(u), \delta(v) \leq 3$, the neighbors of $u$ and $v$ in $H$ are contained in $\{h_k, h_{k+1}, h_{k+2}, h_{k+3}\}$.

Thus, if $u$ and $v$ are not adjacent, then we can choose and independent set of size $r+1$ from the vertices $V(H)-X$, $u$, $v$, and $z$, a contradiction.

*Case 2:* $X$ is a subset of $\{h_1, h_2, h_3, h_4\}$, $k = k' = 1$, and both $u$ and $v$ are adjacent to $h_4$. By Lemma 10.2), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_1$ and at least one of $h_2$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is adjacent to neither $u$ nor $v$. Thus, if $u$ and $v$ are not adjacent, then we can choose and independent set of size $r+1$ from the vertices $V(H)-X$, $u$, $v$, and $z$, a contradiction.

*Case 3:* $X$ is a subset of $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$, $k = k' = 2r-3$, and either $\delta(u) = 3$ or $\delta(v) = 3$. By Lemma 10.3), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_{2r}$ and at least one of $h_1$ or $h_{2r-1}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is adjacent to neither $u$ nor $v$. Thus, if $u$ and $v$ are not adjacent, then we can choose and independent set of size $r+1$ from the vertices $V(H)-X$, $u$, $v$, and $z$, a contradiction.

*Case 4:* The neighbors of $u$ and $v$ on $H$ are identical. By our suppositions and Lemma 5, we have that $\delta(u) = \delta(v) = 3$. If $2 \leq k \leq 2r-4$, then by Lemma 10.1), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to both $h_1$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these two vertices in $H$, and $z$ is adjacent to neither $u$ nor $v$. If $r \geq 4$, then by Lemma 6, $X$ is four consecutive vertices contained in $\{h_2, h_3, \ldots, h_{2r-1}\}$. If $r = 3$, by Lemmas 6, 7, 8, and 9, $X$ is four consecutive vertices contained in either $\{h_2, h_3, \ldots, h_{5=2r-1}\}$ or $\{h_1, h_2, h_3, \ldots, h_6\}$. If $X$ is four consecutive vertices contained in $\{h_2, h_3, \ldots, h_{2r-1}\}$, and $u$ and $v$ are not adjacent, then we can choose and independent set of size $r+1$ from the vertices $V(H)-X$, $u$, $v$, and $z$, a contradiction. If $r = 3$ and $X$ is contained in $\{h_1, h_2, h_3, \ldots, h_6\}$, then $u$ and $v$ are adjacent by Lemma 9.

Likewise, if $k = 1$ and $r \geq 4$, then by Lemma 6, $X$ is contained in $\{h_1, h_2, h_3, h_4\}$. Of course, even if $r = 3$, it may still be the case that $X$ is contained in $\{h_1, h_2, h_3, h_4\}$. If this is true, since $\delta(u) = \delta(v) = 3$ implies each of $u$ and $v$ is adjacent to $h_4$, by Lemma 10.2) there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_1$ and at least one of $h_2$ and $h_{2r}$. Furthermore, $z$ is adjacent to only these vertices in $H$, and $z$ is adjacent to neither $u$ nor $v$. Thus if $u$ and $v$ are not adjacent, then we can choose and independent set of size $r+1$ from the vertices $V(H)-X$, $u$, $v$, and $z$, a contradiction. Therefore, suppose $r = 3$, but $X$ is not contained in $\{h_1, h_2, h_3, h_4\}$. By Lemma 6, $X$ is contained in either $\{h_1, h_2, h_5, h_6\}$, $\{h_1, h_4, h_5, h_6\}$, or $\{h_1, h_2, h_3, h_6\}$. In any event, we can apply Lemmas 7, 8, and 9 to deduce that $u$ and $v$ are adjacent.

If $k = 2r-3$, $X$ is contained in $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$. Since $\delta(u) = \delta(v) = 3$, then by Lemma 10.3), there exists a vertex $z \in V(G) - V(H)$ such that $z$ is adjacent to $h_{2r}$ and at least one of $h_1$ or $h_{2r-1}$. Furthermore, $z$ is adjacent to only these vertices on $H$, and $z$ is adjacent to neither $u$ nor $v$. Thus if $u$ and $v$ are not adjacent, then we can choose an independent set of size $r+1$ from the vertices $V(H)-X$, $u$, $v$, and $z$, a contradiction.

Finally, if $k > 2r-3$, $X$ is three or less consecutive vertices, which we already showed implies $u$ is adjacent to $v$. $\square$

## Open Problems

Analogous to the definitions of path number and bipartite number, the *tree number* of a graph $G$ is the maximum order of an induced tree subgraph. Likewise, the *induced circumference* of $G$ is the maximum order of an induced cycle subgraph. These invariants are denoted by $t = t(G)$ and $C_{\text{ind}} = C_{\text{ind}}(G)$, respectively. Let $\kappa = \kappa(G)$ be the connectivity of $G$. The following conjecture of Graffiti.pc interested us because of its similarity to the well-known Erdös-Chvátal Theorem, which states that if $\kappa \geq \alpha - 1$ for a graph $G$, then the graph has a Hamiltonian path.

**Conjecture 3.** *(Graffiti.pc 199) Let $G$ be a graph. If $\kappa \geq t - 2$, then $G$ contains a Hamiltonian path.*

Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a graph $G$ arranged in non-decreasing order. The *annihilation number* of $G$, $A = A(G)$, is the largest integer $k$ such that the sum of the first $k$ terms of the sequence, $d_1 + d_2 + \ldots + d_k$, is at most half the sum of the entire sequence (i.e. the size of $G$). This invariant was introduced in [13], where it was shown to be an upper bound on the independence number of the graph. The definition presented here is due to Fajtlowicz, although R. Pepper showed it was equivalent to the original definition presented in [13].

**Conjecture 4.** *(Graffiti.pc 205) Let $G$ be a graph. If $C_{ind} \geq 2(A - 1)$, then $G$ contains a Hamiltonian path.*

**Conjecture 5.** *(Graffiti.pc 201) Let $G$ be a graph. If $p = n - d_2 + 1$, then $G$ contains a Hamiltonian path.*

For a graph $G$, let $L = L(G)$ denote the maximum number of leaves of a spanning tree of $G$. We call this invariant the *leaf number* of $G$. The following conjecture of Graffiti.pc related to $L$ is reminiscent of Dirac's famous sufficient conditions for a graph to contain a Hamiltonian cycle or path. Let $\delta = \delta(G)$ be the minimum degree of $G$.

**Conjecture 6.** *(Graffiti.pc 190) Let $G$ be a graph. If $\delta \geq \dfrac{L+1}{2}$, then $G$ contains a Hamiltonian path.*

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