Graffiti.pc on the Independent-Domination Number of a Graph

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Abstract

The independent-domination number of a graph is the cardinality of a smallest set of mutually non-adjacent vertices which has the property that every vertex not in the set is adjacent to at least one that is. We present several conjectures made by the computer program Graffiti.pc about the independent-domination number of graphs, providing proofs and partial results for some of these conjectures.

keywords: independent-domination, independence number, maximum degree vertices, large degree vertices, Graffiti.pc.

1 Definitions and Introduction

Given a finite simple graph $G = (V, E)$, an *independent set* is a subset of V such that no pair of vertices in the subset are adjacent. A *dominating set* is a subset of V such that every vertex not in the subset is adjacent to at least one vertex that is. The *independence number*, $\alpha = \alpha(G)$, is the cardinality of a largest independent set. The *domination number*, $\gamma =$

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$\gamma(G)$, is the cardinality of a smallest dominating set. The *independent-domination number*, $i = i(G)$, is the cardinality of a smallest independent and dominating set. For $S \subseteq V$, the subgraph induced by S is denoted $[S]$. For graph theory terms and definitions not explicitly described or defined below, the reader is referred to any basic graph theory text.

The Graffiti.pc conjecture-making program was written by E. DeLaViña and inspired by a related program called Graffiti which was written by S. Fajtlowicz. Some details of these programs can be found in [1] and [2], and here we simply note that the programs' conjectures take the form of upper and lower bounds for a user selected graph invariant over a user selected graph property.

In recent years, the Graffiti.pc program has been queried for bounds on invariants related to certain kinds of dominating subsets for connected graphs. From 2007 through 2009, we settled many Graffiti.pc conjectures about total domination number and some of these results can be found in [3], [4] and [5]. In 2010, we worked primarily on conjectures about 2-domination number, and some of these results can be found in [6], [7] and [8]. In late 2010, Graffiti.pc was queried for conjectures about the independent-domination number for connected graphs. In the paper at hand we present some results on those conjectures. For the full list of Graffiti.pc conjectures and their current status see [9].

2 Main Results

Two of the program's conjectures (numbered 419a and 419b in [9]) inspired and now follow from the following simple theorem. It gives a slight improvement on the inequality, $i(G) \leq \alpha(G)$, whenever there is a vertex in every maximum independent set of the graph G.

Theorem 2.1. *If the intersection of all maximum independent sets of a connected graph G is non-empty, then*

$$i(G) \leq \alpha(G) - 1 < \alpha(G).$$

Proof. Let G be a connected graph and suppose the intersection of all maximum independent sets of G is non-empty. Let v be a vertex in every maximum independent set and let I be a maximum independent set. Denote by N the neighbors of v. Let J be an independent-dominating set of $[N]$ and let K be the set of vertices of $I - \{v\}$ which have neighbors in J. Now, starting with the set $I - \{v\} + J - K$, use a greedy algorithm to form an independent-dominating set of G, and call this set I^*. Observe that I^* is not a maximum independent set since it does not contain v. Hence, $|I^*| < |I|$ and consequently, $i(G) < \alpha(G)$, proving the theorem.

$\square$

Remark 2.2. *The proof above actually establishes something stronger than the statement of the theorem. Namely, it shows that, for any vertex in every maximum independent set, there exists an independent-dominating set that does not contain that vertex. The theorem above is then a corollary to this statement.*

The following two conjectures of Graffiti.pc, Conjecture 2.3 and Conjecture 2.7 (but numbered 422a and 422c in [9]), remain open in general. We provide some partial results and discussion after stating each conjecture. The similarity between them perhaps explains why our partial results also have much in common.

Conjecture 2.3. *Let G be a graph and let M be the set of vertices with maximum degree. Then,*

$$i(G) \leq \alpha([V - M]) + \frac{2}{3} m([M]),$$

where $[V - M]$ denotes the subgraph induced by the vertices without maximum degree and $m([M])$ denotes the number of edges in the subgraph induced by M.

As a starting point, it seemed natural to focused on the special case when M is independent. In this case, the statement simplifies as follows.

Conjecture 2.4. *Let G be a graph and let M be the set of vertices with maximum degree. If M is an independent set, then*

$$i(G) \leq \alpha([V - M]),$$

where $[V - M]$ denotes the subgraph induced by the vertices without maximum degree.

Our next result is a special case of 2.4.

Theorem 2.5. *Let G be a graph and let M be the set of vertices with maximum degree. If M is an independent set and $|M| \leq 3$, then*

$$i(G) \leq \alpha([V - M]),$$

where $[V - M]$ denotes the subgraph induced by the vertices without maximum degree.

Proof. Suppose that there is a unique vertex of maximum degree. In this case, construct a maximal independent set in $[V - M]$ which contains a neighbor of the unique maximum degree vertex (using a greedy algorithm). This set will be independent-dominating in G and have order at most $\alpha([V - M])$.

Suppose next that there are exactly two vertices of maximum degree. Let a and b be the two maximum degree vertices. Denote the neighborhoods of a and b by A and B respectively (note that under our hypothesis of M being independent, none of the vertices in $A \cup B$ are themselves of maximum degree). Now, if there is a vertex $x \in A \cap B$, then build a maximal independent set of $[V - M]$ containing x (using a greedy algorithm). This set will be an independent-dominating set of G and have order at most $\alpha([V - M])$. Thus, we may assume $A \cap B = \emptyset$. Let $x \in A$. Since x has at most $\Delta - 2$ neighbors in $[V - M]$ and $|B| = \Delta$, there is a vertex $y \in B$ which is not adjacent to x. Build a maximal independent set of $[V - M]$ which contains both x and y (using a greedy algorithm). This set will be an independent-dominating set of G and have order at most $\alpha([V - M])$.

Finally, suppose that $|M| = 3$. Let a, b, and c be the three maximum degree vertices with neighborhoods A, B, and C respectively (note that under our hypothesis of M being independent, none of the vertices in $A \cup B \cup C$ are themselves of maximum degree). If there is a vertex in the intersection of all three of these neighborhoods, or if any pair of these neighborhoods intersect but miss the third, then the proof follows along the same lines as the previous two cases. Thus, we may assume that these sets are pairwise disjoint.

Let $x \in A \cup B \cup C$ be a vertex which has the most neighbors in one of the other sets. Without loss of generality, we may assume $x \in A$ and B is the set which realizes this maximum number of external neighbors. Denote by $A(z)$, $B(z)$, and $C(z)$ the sets of neighbors a vertex z has in sets A, B, and C respectively. Since x can have at most $\Delta - 2$ neighbors in $A \cup B \cup C$, there is a vertex $y \in B$ which is not adjacent to x. It now remains to show there is a vertex $w \in C$ which is adjacent to neither x nor y. To that end, observe the following inequalities;

$$|C(x)| \leq \Delta - 2 - |B(x)|$$

$$|C(y)| \leq |B(x)|.$$

The second of these is true by the way we selected x. Together, they yield the inequality,

$$|C(x)| + |C(y)| \leq \Delta - 2 < \Delta = |C|.$$

This shows that there is a vertex $w \in C$ adjacent to neither x nor y. Now, build a maximal independent set of $[V - M]$ which contains x, y, and w (using a greedy algorithm). This set will be an independent-dominating set of G and have order at most $\alpha([V - M])$. This completes the proof. □

Remark 2.6. *Theorem 2.5 is actually true when M is not independent as well, but the details seemed to be too much of a distraction. The proof technique we used fails when $|M| \geq 4$, though, in spite of that, we have found no counter-example to Conjecture 2.3.*

Conjecture 2.7. *Let G be a graph of order n and let L be the set of vertices with degree more than $\frac{n}{2}$. Then,*

$$i(G) \leq \alpha([V - L]) + \frac{2}{3}\Delta([L]),$$

where $[V - L]$ denotes the subgraph induced by the vertices with degree at most $\frac{n}{2}$ and $\Delta([L])$ denotes the maximum degree of the subgraph induced by L.

Again, it seemed natural to focused on the special case when L is an independent set. In this case, the statement simplifies as follows.

Conjecture 2.8. *Let G be a graph of order n and let L be the set of vertices with degree more than $\frac{n}{2}$. If L is an independent set, then*

$$i(G) \leq \alpha([V - L]),$$

where $[V - L]$ denotes the subgraph induced by the vertices with degree at most $\frac{n}{2}$.

Before proceeding, observe that if L is empty, the bound follows trivially, since $\alpha([V - L]) = \alpha(G) \geq i(G)$.

Lemma 2.9. *Let G be a graph of order n and let L, as defined in Conjecture 2.7, be a non-empty independent set. If there is a vertex v adjacent to all vertices of L or all but one vertex of L, then*

$$i(G) \leq \alpha([V - L]).$$

Proof. Suppose v is adjacent to all vertices of L. Use any greedy algorithm to build a maximal independent set of $[V - L]$ containing v. This set will be an independent-dominating set of G and have order at most $\alpha([V - L])$, which completes the proof.

Suppose v is adjacent to all but one vertex of L. Suppose $x \in L$ is not adjacent to v. Since the degree of x is more than $\frac{n}{2}$ and v could not be in L by our hypothesis, there is at least one vertex w adjacent to x which is not adjacent to v. Use any greedy algorithm to build a maximal independent set of $[V - L]$ containing both v and w. This set will be an independent-dominating set of G and have order at most $\alpha([V - L])$, which completes the proof. □

Theorem 2.10. *Let G be a graph of order n and let L be the set of vertices with degree more than $\frac{n}{2}$. If L is an independent set and $|L| = k \leq 4$, then*

$$i(G) \leq \alpha([V - L]),$$

where $[V - L]$ denotes the subgraph induced by the vertices with degree at most $\frac{n}{2}$.

Proof. In light of Lemma 2.9, we may assume that each of the $n-k$ vertices in $V-L$ is adjacent to at most $k-2$ vertices of L. However, since each of the k vertices of L is adjacent to more than $\frac{n}{2}$ vertices of $V-L$, we get the following lower and upper bounds on the number of edges between L and $V-L$, denoted below as m':

$$\frac{kn}{2} < m' \leq (n-k)(k-2). \tag{1}$$

It turns out that this inequality can only be true when $k > 4$. Consequently, Conjecture 2.7 is true when $k \leq 4$ because Lemma 2.9 will apply. □

Corollary 2.11. *Let G be a graph of order n and let L, as described in Conjecture 2.7, be an independent set. If $k = |L| > 4$, then Conjecture 2.7 is true whenever,*

$$n \leq \frac{2k^2-4k}{k-4}. \tag{2}$$

Proof. When $k > 4$, Inequality 1, in the proof of the above theorem, can be solved for n to get the values for which Lemma 2.9 does not apply. For all smaller values of n – those prescribed by Inequality 2 in the statement of this corollary – the result follows from Lemma 2.9. □

Remark 2.12. *Theorem 2.10 is actually true when L is not independent as well, but the details seemed to be too much of a distraction. The proof technique we used fails when $|L| \geq 5$, though, in spite of that, we have found no counter-example to Conjecture 2.7.*

3 Other Conjectures

For this query on independent-domination, Graffiti.pc reported a few dozen conjectures. To date several have been refuted. In addition to Conjectures 2.3 and 2.7, which remain open, we present the following open conjectures.

Conjecture 3.1. *Let G be a graph of order n and let M be the set of vertices of maximum degree. Then*

$$i(G) \leq n - \gamma([N(M)]),$$

where $[N(M)]$ denotes the subgraph induced by neighbors of the maximum degree vertices.

The *degeneracy* of a graph G, denoted $d(G)$, is the maximum of minimum degrees of all induced subgraphs of G. The *center* of a graph G, denoted by C, is the set of all vertices of minimum eccentricity of G. The distance from a vertex v to a set of vertices is the smallest distance from v to any of the vertices in the set. The *eccentricity of the center*, denoted by $ecc(C)$, is the maximum distance from the center to vertices not in the center. Note that if the center is a single vertex, then the eccentricity of the center is equal to what is commonly called the radius of the graph.

Conjecture 3.2. *Let G be a connected graph and let C be the center of G. Then*

$$i(G) \leq d(G^c) - ecc(C) + 2,$$

where G^c is the complement graph of G.

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