Methods of Proofs

Recall we discussed the following methods of proofs:

- Vacuous proof
- Trivial proof
- Direct proof
- Indirect proof
- Proof by contradiction
- Proof by cases.

A **vacuous proof** of an implication happens when the hypothesis of the implication is always false.

Example 1: Prove that if x is a positive integer and $x = -x$, then $x^2 = x$.

An implication is **trivially true** when its conclusion is always true.

A declared mathematical proposition whose truth value is unknown is called a **conjecture**.

One of the main functions of a mathematician (and a computer scientist) is to decide the truth value of their claims (or someone else's claims).

If a conjecture is proven true we call is a *theorem*, *lemma* or *corollary*; if it proven false, then usually discarded.

A **proof** is a sequence of statements bound together by the rules of logic, definitions, previously proven theorems, simple algebra and axioms.

Definition: An integer n is even if there exists an integer k such that $n = 2k$. An integer n is odd if there exists an integer k such that $n = 2k + 1$.

Example: Use the definition of odd to explain why 9 is odd, but why 8 is not odd.
Axiom (Closure of addition over the integers): If a and b are integers, then $a + b$ is an integer.
Axiom (Closure of multiplication over the integers): If a and b are integers, then $a \cdot b$ is an integer.

Example 2: (fill in the blanks)

(i) Property of Closure of ♦ over the set of numbers S:
If a and b are _____________, then ____ is ___________.

(ii) True or False: The integers have closure with respect to subtraction.

(iii) True or False: The natural numbers have closure with respect to subtraction.

(iv) True or False: The integers have closure with respect to division.

(v) True or False: The real numbers have closure with respect to division.

(vi) True or False: The nonzero real numbers have closure with respect to division.

Example 3:
i. Write the proposition “the product of two irrational numbers is irrational“ in symbolic logic notation.

ii. Prove or disprove that the product of two irrational numbers is irrational.

Example 4: Lemma 1. If n is even, then n^2 is even.

i. Write the proposition in symbolic logic notation.

ii. Write the contrapositive of the implication in symbolic logic notation

iii. Proof:

Example 5: Lemma 2. If n^2 is even, then n is even.

i. Write the proposition in symbolic logic notation (with the necessary quantifiers).

ii. Proof:

Theorem 1: An integer n is even if and only if n^2 is even.
Proof: If n is even, then n^2 is even is true by Lemma 1. The converse, if n^2 is even, then n is even is true by Lemma 2. Hence the biconditional statement n is even if and only if n^2 is even is true.

Example 6: Prove that the sum of two odd integers is even. i.e. If p and q are odd integers, then $p + q$ is an even integer.

i. Write the proposition in symbolic logic notation.

ii. Proof:

Summary. If we are proving $p \rightarrow q$, then

A **direct proof** begins by assuming p is true. : : until we conclude q.	An **indirect proof** begins by assuming $\sim q$ is true. : : until we conclude $\sim p$.

An example of a proof by contradiction.

Example 7: Prove that $\sqrt{2}$ is irrational.

Proof: Assume by way of contradiction that can be represented as a quotient of two integers p/q with $q \neq 0$. Further, we assume that p/q is in lowest terms, i.e. we assume that

The integers p and q have no common factor. (1)

Thus, by assumption $\sqrt{2} = p/q$, and now squaring both sides yields

$$2 = \frac{p^2}{q^2} \quad \text{or} \quad p^2 = 2q^2 \qquad (2)$$

This implies that p^2 is even, and by Theorem 1, p must also be even. So we write $p = 2k$ for k some integer, substitute into the second equation of (2), and by cancellation we see that

$$q^2 = 2k^2. \qquad (3)$$

This says that q^2 is even, and again by Theorem 1, q must also be even. From statements (2) and (3), it follows that

p and q both have 2 as a common factor. (4)

Statements (1) and (4) are contradictory. Thus, $\sqrt{2}$ is not a rational fraction. □

Summary. If we are proving $p \rightarrow q$, then

A **direct proof** begins by assuming p is true. : : until we conclude q.	An **indirect proof** begins by assuming $\sim q$ is true. : : until we conclude $\sim p$.	An **proof by contradiction** begins by assuming $p \wedge \sim q$ is true. : : until we reach a contradiction

Example 8: Prove that if $3n + 2$ is odd, then n is odd.

i. Write the proposition in symbolic logic notation.

ii. Write the negation of the proposition in symbolic logic notation.

iii. Proof:

Definition. Let x be a real number. Then $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$.

An example of a proof by cases:

Example 9: Prove if x is a real number, then $|-x| = |x|$.

Definition. A function $f{:}A \rightarrow B$ is *one-to-one* if and only if

$$\forall x \forall y\, (f(x) = f(y) \rightarrow x = y),$$

which is logically equivalent to its contrapositive

$$\forall x \forall y\, (x \neq y \rightarrow f(x) \neq f(y)).$$

Example 10: Prove that the real valued function $f(x) = x + 1$ is one-to-one.

Example 11: Prove the following statements about an integer x are equivalent.

(i) $3x+2$ is even

(ii) $x+5$ is odd

(iii) x^2 is even