Bayesian Analysis of the Sequential Inspection Plan via the Gibbs Sampler

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A complex product, such as a software system, is often inspected more than once in a sequential manner to further improve its quality and reliability. In such a case, a particularly important task is to accurately estimate the number of errors still remaining in the product after a series of multiple inspections. In the paper, we first develop a maximum likelihood method of estimating both the number of undiscovered errors in the product and the detection probability. We then compare its performance with that of an existing estimation method that has several limitations. We also propose a Bayesian method with noninformative priors, which performs very well in a Monte Carlo simulation study. As the prior knowledge is elicited and incorporated in the analysis, the prediction accuracy of the Bayesian method improves even further. Thus, it would be worthwhile to use various estimation methods and compare their estimates in a specific inspection environment.

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1. Introduction

Suppose that a certain complex product, such as an automobile, a mobile home, or a software system, has an unknown number $N$ of defects, errors, faults, or nonconformities. Because of inspection errors, the product will be inspected more than once in a sequential manner to further improve its quality and reliability. After a series of $k$ inspection cycles, the inspection results will be represented as a set $x = \{x_1, x_2, \ldots, x_k\}$, where $x_i$ is the numbers of defects discovered and removed during the $i$th inspection cycle. In the paper, we consider the problem of estimating, based solely on the inspection results $x$, (i) the number of undiscovered errors (or, equivalently, the total number of errors $N$ initially contained in the product) and (ii) the inspector’s unknown detection probability $p$.

Estimating the number of errors still remaining in the product is an important task in reliability engineering. In software reliability, for example, the accurate estimation of the number of undiscovered errors “not only helps certify the application-readiness of the software, but also provides an indication of the effort that will be needed for customer support and for the upgrading of future program releases” (Jewell 1985a, p. 663). A similar interpretation arises in proof-reading a manuscript for typographical errors, inspecting a new home for construction flaws (Bonett and Woodward 1994), or screening a production lot for quality assessment and assurance.

Estimating the inspection effectiveness is another important task in quality management. For example, the airport security authority is interested not only in the number of illegal items not detected during the inspection process, but also in the detection probability of the inspection procedure. Based on the inspection effectiveness, we may determine the number of inspections required to achieve a desired level of product quality (Greenberg and Stokes 1995).

The sequential inspection model considered in the paper should be contrasted with the parallel inspection model, in which several inspectors are put to work independently on identical copies of the product, secretly identifying, but not removing, the defects that they find. In such a case, some defects discovered by one inspector could be also detected by other inspectors. In the sequential inspection plan, on the other hand, any defects discovered during an inspection cycle are removed or corrected so that they will not leak through the subsequent inspection cycles. Thus, the same defects will not be discovered more than once in the sequential inspection process.

Note that the sequential inspection plan can be classified further into the continuous-time case and the discrete-time case. In the continuous-time case, faults are discovered one after another in random order, and the inter-discovery time between two consecutive faults is assumed to be a continuous random variable, usually described as a non-homogeneous Poisson process. In the software reliability growth model, for example, the inspection history is expressed as a set $t = \{t_1, t_2, \ldots, t_k\}$, in which $t_i$ is the time between the discoveries of the $(i-1)$th and $i$th faults. In the discrete-time version of the sequential inspection model, the number of faults $x_i$ detected during the $i$th inspection cycle is revealed at the end of the discrete time period. Thus, the inspection results are summarized as a
set $x = \{x_1, x_2, \ldots, x_k\}$, where $x_i$ is the numbers of defects detected during the $i$th inspection cycle.

In this paper, we focus on the discrete-time version of the sequential inspection plan. We first develop two novel methods of estimating both $N$ and $p$, and then compare their performances with that of an existing model. In the next section, we review several articles related to those multiple inspection models, including the existing nonlinear regression method. We then propose two new estimation methods: the maximum likelihood method in §3 and the Bayesian method in §4. A numerical example is given in §5 to illustrate these estimation methods. In §6, the prediction accuracies of the new methods are compared with that of the existing method in an extensive Monte Carlo simulation. Section 7 contains concluding remarks on the possible applications of and extensions to our proposed models.

2. Literature Review

2.1. Parallel Inspection Plan

Most of the past research has been focused on the parallel inspection plan. One of the widely used techniques is a class of statistical methods known as “capture-recapture models” (Briand et al. 2000, Bonett 1988). This technique is most commonly used by biologists in estimating the size of wildlife populations in wildlife research. Suppose that animals are captured, marked, and released on several trapping occasions. If an animal bearing a mark is captured on a subsequent trapping occasion, it is said to be “recaptured.” Based on the number of marked animals that are recaptured, we can estimate the population size $N$. When many marked animals are recaptured, for example, we can argue that the population size $N$ is small, and vice versa.

In recent years, capture-recapture methodology has been given to new applications that address problems far different from those traditionally considered. Included among them is the problem of estimating the number of undetected errors in the parallel inspection plan. If there are two inspectors, for example, the inspection results can be summarized as $x = \{x_1, x_2, x_{12}\}$, where $x_i$ is the number of defects discovered by inspector $i$ and $x_{ij}$ is the number of defects found by both inspectors $i$ and $j$. Then, the total number of defects $N$ can be estimated as:

$$\hat{N} = \frac{n_1 \times n_2}{n_{12}},$$

which is widely known as the Lincoln-Peterson estimator (Seber 1986).

This estimator is not suitable if inspectors are dependent or the detection probabilities are not homogeneous. Bonett et al. (1986) developed a nonlinear regression model that accommodates both restrictions. Readers are referred to Pollock (1991) for more details about the modeling of capture, recapture, and removal statistics in the estimation of demographic parameters of fish and wildlife populations, and to Briand et al. (2000) for a comprehensive analysis of capture-recapture models in the field of software inspections.

Since information about $N$ may be regarded as being updated from one sampling occasion to the next in capture-recapture surveys, several Bayesian formulations have been also developed. For example, Castledine (1981) showed that the prior knowledge can greatly affect inferences about the population size. In the context of the parallel inspection plan, Jewell (1985a) proposed a Bayesian model in which $N$ is generated by a Poisson process and $p$ is beta distributed with $(a, b)$. The Poisson parameter $\lambda$ is further assumed to be a random quantity from a gamma distribution with $(\alpha, \beta)$.

Unfortunately, the possible difficulty of calculating posterior quantities of interest in Bayesian analysis is sometimes overcome by introducing unwarranted assumptions. As an alternative to both analytical calculation and numerical approximation, George and Robert (1992) showed how Gibbs sampling can be applied to calculate the Bayes estimates in the capture-recapture setting. Recently, Basu and Ebrahimi (2001) extended the Bayesian model so that it could incorporate, via the Gibbs sampling, the dependency among the inspectors and the nonhomogeneous detection probabilities in the parallel inspection plan.

2.2. Sequential Inspection Plan

The continuous-time version of the sequential inspection problem has also been approached from the Bayesian point of view. For example, Jewell (1985b) developed a Bayesian model in which the prior distribution of $N$ is expressed as the negative binomial distribution and the unit failure rate is gamma distributed. A unified approach to the nonhomogeneous Poisson process in software reliability models is given by Kuo and Yang (1996). Readers are referred to Bergman and Xie’s (1991) expository article for a comprehensive analysis of continuous-time reliability models and their Bayesian extensions.

Little consideration has been given to the discrete-time version of the sequential inspection plan in the literature. In an article published in Management Science, Bonett and Woodward (1994) proposed a nonlinear regression model for the problem of estimating both $N$ and $p$ based on $x = \{x_1, x_2, \ldots, x_k\}$. In the sequential inspection plan, the defects found during the $i$th inspection have obviously not been detected during the previous $(i - 1)$ inspections. Thus, the expected value of $X_i$ is

$$E[X_i] = \mu_i = Nq^{i-1}p \text{ for } i = 1, 2, \ldots, k,$$

where $q = 1 - p$.

The expected value in (2) can be rewritten as

$$\mu_i = \exp[\ln N + \ln p + (i - 1) \ln q]$$

for $i = 1, 2, \ldots, k$. (3)
Thus, the inspection results can be expressed in a matrix
form as follows:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
\vdots \\
X_k
\end{bmatrix} = \exp \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
1 & 1 & 2 \\
\ddots & \ddots & \ddots \\
1 & 1 & k-1
\end{bmatrix} \begin{bmatrix}
\ln N \\
\ln p \\
\ln q \\
\vdots \\
\ln \epsilon_k
\end{bmatrix},
\]

where \( \epsilon_i \) is an independent random error term for the \( i \)th inspection.

To obtain an estimator of the regression parameters \( \{\ln N, \ln p, \ln q\} \) in the nonlinear regression model in (4), Bonett and Woodward (1994) proposed to use the method of “minimum modified transformed chi-square,” which is an iterative method with the initial solutions, \( \{\ln s_1, \ln(x_1/s_1), \ln(1-x_1/s_1)\} \), where \( s_1 = x_1 + x_2 + \cdots + x_i \). In our numerical analysis, we found that the convergence is very fast, although the iterative method often fails to provide sensible estimates, particularly when the true values of \( N \) and \( p \) are relatively small.

In many practical situations, the inspector may have personal or subjective knowledge about \( N \) and \( p \) prior to inspection. Thus, there is no reason not to cast the discrete-time sequential inspection problem in a Bayesian framework. Rallis and Lansdowne (2001) appear to be the first who developed a Bayesian model for the discrete-time sequential inspection plan. They assumed that (i) the number of defects \( N \) is expressed as a Poisson distribution with the mean \( \lambda \) and (ii) the detection probability \( p \) is a known constant.

The Bayesian model developed in the paper is different from Rallis and Lansdowne’s (2001) model in those assumptions. First, we assume that the number of defects \( N \) has a negative binomial prior with parameters \( \alpha \) and \( \beta \), which we believe has many desirable properties. Specifically, by changing the parameter values \( \alpha \) and \( \beta \), we can represent prior distributions with different location, dispersion, shape, and the like, including not only the Rallis and Lansdowne (2001) Poisson prior as a special case, but also the noninformative or vague prior of \( N \).

Second, we may have some prior knowledge about the unknown detection probability \( p \) in practice. Thus, we assume that the prior density for \( p \) is expressed in the paper as a beta distribution with parameters \( a \) and \( b \), which can represent a wide variety of states of prior information about \( p \). For example, if both \( a \) and \( b \) approach infinity, while the ratio \( a/b \) being kept constant, then the prior distribution represents the Rallis and Lansdowne (2001) case in which \( p \) is a known constant. If \( a = 1 \) and \( b = 1 \), on the other hand, the beta prior becomes the noninformative, “flat” prior.

In the paper, we will demonstrate how Gibbs sampling can be used to calculate the posterior marginal densities of both \( N \) and \( p \) for the Bayesian model with the negative binomial prior for \( N \) and the beta prior for \( p \). Throughout the remainder of the paper, we will simply use the term “defects” to refer to errors, bugs, nonconformities, or faults in a complex product or system, for which the inspectors may use visual or machine-aided inspection techniques. The term “inspections” implies a series of test, detection, and correction cycles.

In the next section, we develop a simple method of maximum likelihood as a benchmark for measuring the performance of various sequential inspection models.

### 3. Maximum Likelihood Method

#### 3.1. Likelihood Function

Suppose that the total number of defects in a given product is \( N (0 \leq N < \infty) \), which is unknown to the inspector. The product will be inspected \( k \) times in a sequential manner for the purpose of detecting those defects. In the presence of inspection error, let \( p (0 < p < 1) \) denote the unknown probability that a certain defect will be detected during an inspection cycle, if not detected and removed earlier. As in Bonett and Woodward (1994) and Rallis and Lansdowne (2001), we assume that the detection probability \( p \) remains the same across inspection cycles. In such a case, the sequential inspection plan is also called the repetitive inspection plan (Greenberg and Stokes 1995).

Let \( x_i \) be the number of defects that have been detected during the \( i \)th inspection. For notational convenience, let \( s_i (=x_1 + x_2 + \cdots + x_i) \) be the cumulative number of defects that have been detected and removed during the \( i \) inspections. We assume that all defects discovered during one inspection are corrected before the next inspection begins, but without injecting any new defects during the correction process. Thus, after a series of \( (i-1) \) inspections, there are \( (N-s_{i-1}) \) defects still remaining undetected in the product, and the number of defects \( X_i \), that will be discovered during the \( i \)th inspection is a random sample from the binomial distribution:

\[
P[X_i \mid N-s_{i-1}, p] = \binom{N-s_{i-1}}{x_i} p^{x_i} q^{N-x_i},\]

for \( x_i = 0, 1, \ldots, N-s_{i-1} \),

where \( q = 1 - p \).

After a series of \( k \) inspections, the likelihood function of the inspection history \( X = \{X_1, X_2, \ldots, X_k\} \) is

\[
P[X \mid N, p] = P[X_1 \mid N, p]P[X_2 \mid N-s_1, p] \cdots P[X_k \mid N-s_{k-1}, p]
= \frac{N!}{(N-s_{k})!} \prod_{i=1}^{k} \binom{q^{N-s_i}}{x_i} p^{x_i},
\]

which is a multinomial distribution with \( (k + 1) \) classes. Sometimes it is computationally easier to work with an alternative form (Chun 2005):

\[
P[X \mid N, p] = \frac{N!}{(N-s_{k})!} \prod_{i=1}^{k} x_i! q^{x_i} p^{x_i} (1-p)^{N-x_i}.
\]
This likelihood function will be used to find the maximum likelihood estimators of $N$ and $p$.

3.2. Maximum Likelihood Estimators of $N$ and $p$

Since the method of maximum likelihood has been applied to a great variety of other multiple inspection problems, it is ironic that such a simple method has not been developed for the sequential inspection plan until now. The maximum likelihood estimates of $N$ and $p$ are those values that make the observed data $x = \{x_1, x_2, \ldots, x_k\}$ “most probable” or “most likely.” Rather than maximizing the likelihood function (7) itself, it is easier to maximize its natural logarithm:

$$\ln P[X \mid N, p] = \ln \left[ \frac{N!}{(N-s_k)! \prod_{i=1}^{k} x_i!} \right] + s_k \ln p + \sum_{i=1}^{k} (N-s_i) \ln q. \quad (8)$$

If we calculate the first-order derivative of the log likelihood function with respect to $p$, we obtain the following log likelihood:

$$\hat{p} = \frac{s_k}{\sum_{i=1}^{k} (N-s_{i-1})}. \quad (9)$$

By replacing $p$ in the likelihood function (8) with $\hat{p}$ in (9), we obtain the following log likelihood:

$$\ln P[X \mid N, p] = \ln N! - \ln (N-s_k)!$$

$$+ s_k (\hat{p} \ln \hat{p} + \hat{q} \ln \hat{q}) / \hat{p} + \text{constant}, \quad (10)$$

where $\hat{q} = 1 - \hat{p}$ and $\hat{p}$ is given in (9).

The estimate of $N$ that maximizes the log likelihood in (10) can be easily obtained by any iterative method, such as the SOLVER function in Microsoft Excel. In the next section, we propose another novel method of estimating $N$ and $p$ simultaneously.

4. Bayesian Method

4.1. Prior Distribution of $N$

Suppose that the prior knowledge about the number of defects $N$ can be expressed as a Poisson distribution:

$$P[N \mid \lambda] = \frac{\lambda^N e^{-\lambda}}{N!} \quad \text{for } N = 0, 1, \ldots, \infty, \quad (11)$$

where $\lambda$ has a gamma distribution with parameters $\alpha$ and $\beta$,

$$g(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}. \quad (12)$$

Then, the compound distribution of $N$ is shown to be

$$P[N \mid \alpha, \beta] = \int_{\lambda=0}^{\infty} P[N \mid \lambda] g(\lambda) \, d\lambda$$

$$= \left( \frac{N + \alpha - 1}{N} \right)^\beta \left[ \frac{1}{1 + \beta} \right]^{\alpha + \sum_{i=1}^{k} x_i}. \quad (13)$$

which is a negative binomial distribution with hyper-parameters $\alpha$ and $\beta$. Recall that, for the negative binomial prior, the expected value is $E[N \mid \alpha, \beta] = \alpha\beta$ and the variance is $\text{Var}[N \mid \alpha, \beta] = \alpha\beta(\beta + 1)$. By systematically changing $\alpha$ and $\beta$, we can represent a wide variety of prior distributions with various combinations of the mean and the variance.

Note that the negative binomial prior in (13) includes two limiting cases: the Poisson prior and the noninformative prior. First, suppose that $\alpha\beta$ is kept constant. If $\beta$ approaches 0, then the prior distribution becomes the Poisson distribution, in which both the mean and the variance are equal to $\alpha\beta$. Second, if $\beta$ approaches $\infty$, then the negative binomial prior represents the situation in which the prior knowledge that we have about the parameter $N$ may be very slight and vague when compared with the information about $N$, which we expect to acquire from future inspections.

4.2. Posterior Distribution of $N$

With the negative binomial prior in (13), the marginal distribution of $x$, conditional upon $p$, is given by

$$P[x \mid p] = \sum_{N=0}^{\infty} P[x \mid N, p] P[N]$$

$$= \frac{(\alpha + s_k - 1)!}{(\alpha - 1)! \prod_{i=1}^{k} x_i!} \left( 1 + \beta(1-q^k) \right)^{\alpha}$$

$$\cdot \left[ \prod_{i=1}^{k} \left( \frac{\beta q^{i-1} p}{1 + \beta(1-q^k)} \right)^{s_i} \right], \quad (14)$$

which is known as the negative multinomial or multivariate negative binomial distribution (Johnson et al. 1997, p. 93).

Thus, the posterior distribution of $N$, conditional upon $p$, can be shown to be

$$P[N \mid p, x]$$

$$= \frac{P[x \mid N, p] P[N]}{P[x \mid p]}$$

$$= \left( \frac{\alpha + N - 1}{N - s_k} \right) \left( \frac{\beta}{1 + \beta q^k} \right)^{N - s_k} \left( 1 - \frac{\beta}{1 + \beta q^k} \right)^{s_k}, \quad (15)$$

for $N = s_k, s_k + 1, \ldots, \infty$.

Let $R_k = N - s_k$ be the number of defects still remaining undetected in the product after a series of $k$ inspections. Then, it directly follows from (15) that the posterior distribution of $R_k$ is

$$P[R_k \mid p, x]$$

$$= \left( r_k + s_k + \alpha - 1 \right) \left[ \frac{\beta}{1 + \beta q^k} \right]^{r_k} \left[ 1 - \frac{\beta}{1 + \beta q^k} \right]^{s_k + r_k} \quad \text{for } r_k = 0, 1, \ldots, \infty. \quad (16)$$
which is a negative binomial distribution with parameters 
\((\alpha + s_i)\) and \(\beta q^k/[1 + \beta(1 - q^k)]\).

Consequently, the posterior mean of \(R_k\) is
\[
E[R_k | p, x] = \frac{(\alpha + s_k) \beta q^k}{1 + \beta(1 - q^k)},
\]
and the posterior mean of \(N\) is simply
\[
E[N | p, x] = s_k + E[R_k | p, x] = \frac{(1 + \beta)s_k + \alpha \beta q^k}{1 + \beta(1 - q^k)}. \tag{17}
\]

Consider the two limiting cases of our negative binomial prior. First, for the Poisson prior in which the prior mean \(\alpha \beta\) is a constant \(\lambda\), the posterior distribution in (16) becomes a Poisson with parameters \(\lambda q^k\). With the Poisson prior, the posterior mean of \(N\) in (18) is \(\lambda q^k + s_k\). Second, for the noninformative prior (i.e., \(\beta \rightarrow \infty\)), the posterior distribution of \(R_k\) in (16) is a negative binomial distribution with parameters \(s_k\) and \(q^k/(1 - q^k)\). Thus, the posterior mean of \(N\) with the noninformative prior is given by
\[
\frac{s_k}{(1 - q^k)}, \text{which does not depend on the parameter values } \alpha \text{ and } \beta \text{ of the negative binomial prior in } (13).
\]

As correctly pointed out by a reviewer, the posterior mean of \(N\) in (18) can be also expressed as a weighted average of (i) the prior mean, \(\alpha \beta\), and (ii) the noninformative posterior mean, \(s_k/(1 - q^k)\), as follows:
\[
E[N | p, x] = \alpha \beta w_N + \frac{s_k}{1 - q^k}(1 - w_N), \tag{19}
\]
where the weight \(w_N\) is
\[
w_N = \frac{q^k}{1 + \beta(1 - q^k)}. \tag{20}
\]

### 4.3. Prior Density of \(p\)

The beta distribution has been frequently employed as a conjugate prior distribution for a binomial proportion. Thus, it is natural to take the beta distribution to describe the prior knowledge about the detection probability \(p\):
\[
f(p | a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} p^{a-1} (1-p)^{b-1} \text{ for } 0 \leq p \leq 1. \tag{21}
\]

For the beta prior, the expected value and the variance are
\[
E[p | a, b] = \frac{a}{a + b} \text{ and } \quad \text{Var}[p | a, b] = \frac{ab}{(a + b)^2(a + b + 1)}. \tag{22}
\]

If the ratio, \(a/b\), is kept constant, but both \(a\) and \(b\) are increased, the variance of \(p\) decreases and we can represent the situation in which the prior knowledge is very precise. If \(a = 1\) and \(b = 1\), on the other hand, the beta distribution becomes the uniform or rectangular distribution, which we use in the paper as the noninformative or vague prior for \(p\).

Note that there are other types of noninformative prior, among which one prior could be less informative than another. For example, the Jeffrey’s prior with \(a = 0.5\) and \(b = 0.5\) is shown to be less informative than the uniform or “flat” prior with \(a = 1\) and \(b = 1\) (Zhu and Lu 2004). Berger (1985, p. 90) offered two more noninformative priors for \(p\), but he concluded that “all four possibilities are reasonable.” Gelman et al. (2004, p. 63) also concluded that “in practice, the difference between these alternatives is often small.”

### 4.4. Posterior Density of \(p\)

With the beta prior for \(p\), the marginal distribution of \(x\), conditional upon \(N\), is given by
\[
\begin{align*}
P(x | N) &= \int_{p=0}^1 P(x | N, p) f(p) \, dp \\
&= \frac{N!}{(N-s_i)! \prod_{i=1}^k x_i!} \frac{\Gamma(a + b)}{(a + s_i) \Gamma(b + \sum_{i=1}^k (N - s_i))} \cdot p^{a + \sum_{i=1}^k (N - s_i) - 1} (1 - p)^{b + \sum_{i=1}^k (N - s_i) - 1} \text{ for } 0 \leq p \leq 1, \tag{23}
\end{align*}
\]
which is known as the beta multinomial distribution
(Johnson et al. 1997, p. 31).

Thus, the posterior distribution of \(p\), conditional upon \(N\), can be shown to be
\[
f(p | N, x) = \frac{P(x | N, p) f(p)}{P(x | N)} = \frac{\Gamma(a + b + \sum_{i=1}^k (N - s_{i-1}))}{\Gamma(a + s_i) \Gamma(b + \sum_{i=1}^k (N - s_i))} \cdot p^{a + s_i - 1} (1 - p)^{b + \sum_{i=1}^k (N - s_i) - 1} \text{ for } 0 \leq p \leq 1, \tag{24}
\]
which is the beta distribution with the mean
\[
E[p | N, x] = \frac{a + s_i}{a + b + \sum_{i=1}^k (N - s_{i-1})}. \tag{25}
\]

It is interesting to note that the posterior mean of \(p\) in (25) lies between the prior mean of \(p\) in (22) and the maximum likelihood estimator of \(p\) in (9) as follows:
\[
E[p | N, x] = \frac{a}{a + b} w_p + \frac{s_k}{\sum_{i=1}^k (N - s_{i-1})} (1 - w_p), \tag{26}
\]
where the weight \(w_p\) is
\[
w_p = \frac{a + b}{a + b + \sum_{i=1}^k (N - s_{i-1})}. \tag{27}
\]

### 4.5. Gibbs Estimates of \(N\) and \(p\)

In our Bayesian inspection model, the prior knowledge about \(N\) and \(p\) are expressed as the negative binomial prior \(P[N]\) and the beta prior \(f(p)\), respectively. As shown
in (7), the likelihood function \( P[x | N, p] \) is a multinomial distribution and, unfortunately, the posterior joint density \( P[N, p | x] \) cannot be computed analytically. In such a case, the Gibbs sampler yields a good approximation to the posterior joint density \( P[N, p | x] \) by simulated sampling from the conditional distributions, \( P[N | p, x] \) in (15) and \( f(p | N, x) \) in (24).

The computational process for the Gibbs sampling is illustrated in Figure 1, and the algorithm is described as follow:

- **Input:**
  - Inspection history \( x = \{x_1, x_2, \ldots, x_k\} \).
  - Parameter values \((\alpha, \beta)\) for the negative binomial prior of \( N \), and
  - Parameter values \((a, b)\) for the beta prior of \( p \).

- **Output:**
  - Posterior marginal densities, \( P[N | x] \) and \( f(p | x) \).

- **Procedure:**
  - **Step 1.** Initialization.
    - Set \( s_0 = 0 \).
    - Repeat \( s_j = s_{j-1} + x_j \) for \( j = 1, 2, \ldots, k \).
  - **Step 2.** Specify the initial value, \( p^{(0)} \) and \( N^{(0)} \).
  - **Step 3.** Repeat for \( i = 1, 2, \ldots, i_{\text{max}} \).
    
    \[
    R^{(i)} \sim \text{Negative Binomial} \left( \alpha + s_k, \frac{\beta(1 - p^{(i-1)})^k}{1 + \beta(1 - [1 - p^{(i-1)}]^k)} \right).
    \]
    
    \[
    N^{(i)} = R^{(i)} + s_k.
    \]
    
    \[
    p^{(i)} \sim \text{Beta} \left( a + s_k, b + k \sum_{j=1}^{k} [N^{(i)} - s_j] \right).
    \]

  - **Step 4.** Return the Gibbs sequence, \( p^{(i)} \) and \( N^{(i)} \) for \( i = 1, 2, \ldots, i_{\text{max}} \).

In practice, the initial draws up to a certain point are discarded to allow the effect of the starting value to wear off; this period is referred to as the “burn-in” period. In addition, every \( j \)th draw is stored thereafter to reduce the serial correlation. Provided the sample is large enough, we can estimate any features of the posterior density \( P[N | x] \), or \( f(p | x) \), by taking those samples and forming the relevant sample-based estimates.

For example, the sample average of the sampled draws would be our simulation-based estimate of the posterior mean, while the quantiles of the sampled output would be estimates of the posterior quantiles, with other summaries obtained in a similar manner. Under suitable laws of large numbers (Diebolt and Robert 1994), these estimates would converge to the posterior quantities as the simulation size becomes large.

In the next section, a numerical example is given to illustrate our Bayesian method that employs the Gibbs sampler.

### 5. Numerical Analysis

#### 5.1. Example

Bonett and Woodward (1994) considered the problem of inspecting a mobile home three times, e.g., in the final assembly bay, by the dealer upon delivery, and by the customer at the time of purchase. The number of defects detected and corrected during each of the three inspections is \( x = \{25, 15, 8\} \), respectively. Given the list of inspection results, how can we estimate the number of defects still remaining in the mobile home? This problem is equivalent to estimating the total number of defects \( N \), because the number of defects remain undetected is simply \( N - \sum_{j=1}^{3} x_j \). We may also be interested in estimating the unknown detection probability \( p \).

With the same example, we will illustrate how to estimate \( N \) and \( p \) via the Gibbs sampler and compare the prediction results with those of the maximum likelihood method and the Bonett and Woodward (1994) nonlinear regression method. We programmed all the three estimation methods in the Fortran 90 programming language. Using other available computer software such as Microsoft Excel (Solver) and WinBUGS (Windows operating system version of Bayesian Analysis Using Gibbs Sampling), we double-checked the accuracy of computational results obtained with the Fortran program.

#### 5.2. Prior Distributions of \( N \) and \( p \)

For our Bayesian method, we need to specify the parameter values of the prior distributions. To simulate the situation in which we have very little prior knowledge about the total number of defects \( N \) in the mobile home, we used the “noninformative” or “reference” prior for \( N \). Thus, the posterior distribution in (16) becomes the negative binomial distribution with parameters \( \alpha' = s_k \) and \( \beta' = q^2/(1 - q^2) \). We also assume that the prior knowledge we have about the detection probability \( p \) is very slight and vague. Thus, we set the parameter values \((a, b)\) equal to \((1, 1)\) so that the beta prior for \( p \) becomes the noninformative, “flat” distribution.

Although we use the noninformative priors for illustration, any prior information about \( N \) and \( p \) can be easily...
incorporated into our Bayesian model in practice. The easiest way of subjectively determining parameters ($\alpha$, $\beta$) of the negative binomial prior and ($a$, $b$) of the beta prior is to calculate them from estimated prior moments. A better method of determining prior parameters is to subjectively estimate several quantiles of the prior distribution, and then choose the parameters of the given functional form to obtain a density matching these quantiles as closely as possible. See Press (2003, pp. 70–116) for more details about how to characterize the prior information that we have about $N$ and $p$.

### 5.3. Posterior Distributions of $N$ and $p$

Using the noninformative priors of $N$ and $p$, we generated a Gibbs sequence of $N$ and $p$. We used the random number generators in the IMSL subroutine: `RNNBN` for the negative binomial distribution and `RNBET` for the beta distribution. Setting the initial value $p^{(0)} = 0.5$, we simulated 40,000 draws of $N^{(j)}$ and $p^{(j)}$ in the Gibbs sequence. After a burn-in period of 10,000 draws, we stored every third draw so that 10,000 draws would be used to approximate the posterior distributions of $N$ and $p$.

Figure 2 shows the scatter plot of 10,000 pairs of $N$ and $p$ in the Gibbs sequence. Note that, for given inspection results, the number of undetected errors (or the total number of errors $N$) is estimated to be small if the detection probability $p$ is relatively high, and vice versa. Thus, the joint density of $N$ and $p$ displays a negative relationship between them. From the scatter plot, we can easily obtain the posterior marginal distributions of $N$ and $p$ as shown in Figure 3 and Figure 4, respectively.

### 5.4. Bayesian Inferences of $N$ and $p$

Inference problems concerning $N$ and $p$ can easily be dealt with using Bayesian analysis. The idea is that, since the posterior distributions supposedly contain all the available information about $N$ and $p$, any inferences concerning $N$ and $p$ should consist solely of features of their marginal distributions. The simplest inferential use of the posterior distributions is to report point estimates of $N$ and $p$, with an associated measure of accuracy. Table 1 shows various point estimators and their standard errors when the inspection results are $x = \{25, 15, 8\}$.

For example, the mode, $\hat{N} = 56$, of the marginal distribution in Figure 3 has the interpretation of being the “most likely” value of $N$, given the noninformative priors and the inspection results $x$. Included in Table 1 are other common Bayesian estimators such as the mean and the median of the posterior distributions. The mean is shown to be $\hat{N} = 65.85$, while the median is $\tilde{N} = 60$ for $x = \{25, 15, 8\}$. As stated by Berger (1985, p. 134), it is “worthwhile to calculate and compare all three in a Bayesian study, especially with regard to their robustness to changes in the prior.”

Another common approach to Bayesian inference is to present confidence intervals for $N$ and $p$. The Bayesian analog of a classical confidence interval is called a credible interval. The 90% Bayesian credible intervals for $N$ and $p$ are given in Table 1. In choosing a credible interval, it is usually desirable to try to minimize its size. To do this, one should include in the interval only those points with the largest posterior density, i.e., the “most likely” values of $N$ and $p$. The 90% “highest posterior density” (HPD) credible intervals for $N$ and $p$ are also shown in Table 1.
5.5. Effects of the Prior Variances, \( \text{Var}[N] \) and \( \text{Var}[p] \)

With the same example \( x = \{25, 15, 8\} \), we will show that any prior information on \( N \) and \( p \) could be used to improve the Bayesian estimates. Suppose that the true number of defects is \( N = 60 \) and the detection probability is \( p = 0.4120 \) from Table 1. In the analysis, the prior mean is set equal to \( E[N] = 60 \), while the prior variance has been systematically changed to 120, 240, and 500 to represent various amount of prior information. In addition, we also consider two limiting cases: Poisson prior with \( \text{Var}[N] = 60 \) and the noninformative prior with \( \text{Var}[N] = \infty \). Similarly, the prior variance \( \text{Var}[p] \) is changed from 0.005 to 0.08.

In 10,000 simulation runs, we observed that all the point estimates of \( N \) obtained from the five Bayesian models are very close to the true value \( N = 60 \) for various combinations of \( \text{Var}[N] \) and \( \text{Var}[p] \). However, the accuracy of the predicted values was significantly improved as the prior knowledge about \( N \) and \( p \) became more precise. Figure 5 displays the prediction accuracy measured in terms of the Mean Absolute Percentage Error (MAPE). As expected, the MAPE decreases as the variances of \( N \) and \( p \) decrease (i.e., more prior knowledge about \( N \) and \( p \)).

5.6. Effects of the Prior Mean, \( E[N] \)

With the same example \( x = \{25, 15, 8\} \), we will show that our Bayesian model is robust to “miss-specified” prior

### Table 1. Bayesian estimates of \( N \) and \( p \) when \( x = \{25, 15, 8\} \).

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Number of defects, ( N )</th>
<th>Detection probability, ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mode</td>
<td>Median</td>
</tr>
<tr>
<td>Point estimates*</td>
<td>Estimate</td>
<td>56.00</td>
</tr>
<tr>
<td></td>
<td>Standard error</td>
<td>19.71</td>
</tr>
<tr>
<td>90% interval estimates</td>
<td>Credible interval</td>
<td>51~98</td>
</tr>
<tr>
<td></td>
<td>HPD interval</td>
<td>41~0.57</td>
</tr>
</tbody>
</table>

*Note that the nonlinear regression estimates of \( N \) and \( p \) are 56.5229 and 0.4590, respectively.

...E[N]. Suppose that the detection probability is a known constant \( p = 0.4120 \), while the prior knowledge on \( N \) is expressed as the negative binomial distribution with \( \alpha \) and \( \beta \). In practical situations, we may first find the nonlinear regression or maximum likelihood estimate of \( N \), and then use that information in specifying the prior distribution of \( N \). Thus, it is unimaginable to have the prior mean \( E[N] \) much larger or smaller than the nonlinear regression estimate 59.0914 or the maximum likelihood estimate 56.5229.

For illustration purposes, however, we considered several cases in which the prior mean \( E[N] \) is 40, 50, 70, and 80, respectively. In addition, we also considered the noninformative prior of \( N \). With the noninformative prior of \( N \), the posterior mean is shown to be \( \hat{N} = 60.2493 \) from (18), which is represented as the center line in Figure 6.

As shown in (19), the predicted value of \( N \) is expressed as a weighted average of (i) the “miss-specified” prior mean \( E[N] \) and (ii) the posterior mean with the noninformative prior, \( \hat{N} = 60.2493 \). Suppose that the prior mean is miss-specified as \( E[N] = 80 \). If \( \beta \) approaches zero in Figure 6, then \( N \) is estimated to be 64.2638. If \( \beta \) approaches infinity, then \( N \) is estimated to be 60.2493. For comparison, consider another “miss-specified” prior mean, \( E[N] = 40 \). In such a case, \( N \) is estimated to be somewhere between 56.1319 and 60.2493, depending on the parameter value \( \beta \). The reason why the Bayesian model is robust to the “miss-specified” prior mean is that, as we have more inspection...
results, the Bayesian model puts less weight on the “miss-
specified” prior mean and more weight on the inspection
results.

In the next section, we compare in an extensive Monte
Carlo simulation the performances of the maximum like-
lihood method and the Bayesian method with that of the
nonlinear regression method.

6. Performance Evaluations

6.1. Experimental Design

For all of our Monte Carlo simulations, we assume that
the total number of defects in a given product is \( N = 30 \).
As stated in El Emam and Laitenberger (2001), “this is
a more realistic value for a population size in a software
engineering document.” We assume that, as in Bonett
and Woodward (1994), the software is inspected three
times \((k = 3)\) with the detection probabilities \( \{p_1, p_2, p_3\} \)
for each inspection cycle. In the sequential inspection model
considered in the paper, the detection probabilities are assumed
to be common across all the inspections. Thus, with a
common detection probability \( p \), we first evaluate the per-
fornances of the estimation methods and then test their
sensitivities with respect to the violation of the assumption.

The inspection results \( x = \{x_1, x_2, x_3\} \) were simu-
lated with the random number generator, \( RNMTN \), in
the IMSL Subroutine. The multinomial random numbers,
\( \{x_1, x_2, x_3, 30 - s_3\} \), are distributed with probabilities, \( p_1, q_1p_2, q_1q_2p_3 \), and \( q_1q_2q_3 \), respectively. We repeated the
simulation 10,000 times and calculated the average pre-
cdiction value and the MAPE of each estimation method \( j \)
as follows:

\[
\text{AVERAGE}(\hat{N}_j) = \frac{\sum_{i=1}^{10,000} \hat{N}_{ij}^j}{10,000} \quad \text{and} \\
\text{MAPE}(\hat{N}_j) = \frac{\sum_{i=1}^{10,000} |\hat{N}_{ij} - 30|/30}{10,000}.
\]

Bonett and Woodward’s (1994) nonlinear regression
method often fails to produce the estimates of \( N \) and \( p \).
Thus, we also count the number of occasions in which the
nonlinear regression model fails to provide a stationary
estimate of \( N \).

6.2. Estimation Methods

For the maximum likelihood method, we use the bise-
ction method to find the estimate \( \hat{N} \) that maximizes the
likelihood function. If any inspection result \( x_i \) is zero in
the nonlinear regression method, it was arbitrarily replaced
by a small number, \( 1/k \), as suggested by Bonett
and Woodward (1994). For the Bayesian method, we need to
specify the parameter values of the prior distributions. For
the beta prior of detection probability \( p \), we used a nonin-
formative prior with \( a = 1 \) and \( b = 1 \), which is also called
the “flat” prior. For the negative binomial prior of \( N \), we
used the noninformative prior (i.e., \( \beta \rightarrow \infty \)) for fair com-
parisons with other estimation methods.

To test the effects of the prior knowledge on the pre-
diction accuracy, we further considered several informative
cases. The parameter values are set equal to \((\alpha = 30, \beta = 1)\)
and \((\alpha = 10, \beta = 3)\) so that the priors have the same prior
mean \( \mathbb{E}[N] = 30 \) with different variances. We also consider
the Poisson prior with \( \lambda_0 = 30 \). Thus, we tested a total of
six models in the simulation study. Computations for the
estimation methods have been programmed in Fortran with
IMSL subroutines, which is available upon request.

6.3. Common Detection Probability

We first assume that the detection probabilities are com-
non across all three inspection cycles. By changing the
value \( p \) from 0.3 to 0.9 by 0.1, we simulated 10,000 sets of
inspection results, \( x = \{x_1, x_2, x_3\} \), for each set of parame-
ter values. Table 2 displays the AVERAGE and the MAPE
of the 10,000 predicted values. As shown in Table 2, all
the Bayesian models, including the one with the noninfor-
mative priors of \( N \) and \( p \), consistently produce the point
estimates that are very close to the actual value \( N = 30 \). On
the other hand, the maximum likelihood method tends to
overestimate \( N \) when \( p \) is small. The nonlinear regression
method often fails to converge to stationary values, par-
cularly when \( p \) is small and/or \( N \) is small. Figure 7 shows the
percentage of occasions in which the nonlinear regression
method fails to converge. When \( N = 30 \) and \( p = 0.15 \), for
example, the nonlinear regression method fails to converge
4,043 times or 40.43% of the 10,000 simulation runs.

Figure 8 displays the prediction accuracies measured in
terms of the MAPE. The Bayesian method with noninfor-
mative priors performs better than the maximum likelihood
method, particularly when \( p \) is small. As expected, the per-
fornance of the Bayesian method improves even further as
the prior knowledge about \( N \) becomes more precise.
Besides its nonconvergence problem, the nonlinear regres-
sion method is clearly dominated by the maximum like-
lihood method over the entire range of \( p \).

6.4. Uncommon Detection Probabilities

To test the performance of the prediction models in the
presence of unequal detection probabilities, we systematic-
ally changed \( p_1, p_2, \) and \( p_3 \) from 0.4 to 0.8 by 0.2. Table 3
presents the predicted values of \( N \) where the true value of
\( N \) is 30. As shown in the table, the Bayesian method per-
forms best, even with the noninformative priors of \( N \) and \( p \),
followed by the maximum likelihood method. The non-
linear regression method again fails to converge in many
occasions.

It is interesting to note that all the methods perform
poorly when a less competent inspector (i.e., with a lower
detection probability \( p \)) precedes a more competent inspec-
tor (i.e., with a higher \( p \)). In such a case, both the Bayesian
method and the maximum likelihood method tend to over-
estimate the true value \( N = 30 \), while the regression method
Table 2. Average predicted values of \( N \) and their percentage errors in 10,000 simulation runs when the exact number of defects is \( N = 30 \) and the detection probabilities \( (p_1, p_2, p_3) \) are equal.

<table>
<thead>
<tr>
<th>Detection probability</th>
<th>Test statistics</th>
<th>Bayesian model with ( (\alpha, \beta) )</th>
<th>Maximum likelihood</th>
<th>Regression model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0.9, 0.9, 0.9) )</td>
<td>Average</td>
<td>30.0616, 30.062, 30.062, 30.0621</td>
<td>29.9732, 30.6929</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>0.1106, 0.1111, 0.1113, 0.1116</td>
<td>0.0269, 0.7203</td>
<td></td>
</tr>
<tr>
<td>( (0.8, 0.8, 0.8) )</td>
<td>Average</td>
<td>30.1809, 30.1883, 30.1922, 30.1966</td>
<td>29.7977, 30.7948</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>0.4804, 0.4896, 0.4947, 0.5005</td>
<td>0.2490, 0.9708</td>
<td></td>
</tr>
<tr>
<td>( (0.7, 0.7, 0.7) )</td>
<td>Average</td>
<td>30.3028, 30.3453, 30.3745, 30.4151</td>
<td>29.5330, 31.2076</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>0.9603, 1.0145, 1.0496, 1.0968</td>
<td>0.8757, 1.6698</td>
<td></td>
</tr>
<tr>
<td>( (0.6, 0.6, 0.6) )</td>
<td>Average</td>
<td>30.3658, 30.4903, 30.5977, 30.8209</td>
<td>29.5163, 31.9114</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>1.4436, 1.6203, 1.7554, 2.0072</td>
<td>1.8527, 2.8407</td>
<td></td>
</tr>
<tr>
<td>( (0.5, 0.5, 0.5) )</td>
<td>Average</td>
<td>30.0775, 30.2482, 30.4522, 31.1614</td>
<td>30.1445, —</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>1.8330, 2.2066, 2.5244, 3.3554</td>
<td>3.9624, —</td>
<td></td>
</tr>
<tr>
<td>( (0.4, 0.4, 0.4) )</td>
<td>Average</td>
<td>29.6741, 29.6961, 29.8727, 31.3093</td>
<td>33.9589, —</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>1.9748, 2.6545, 3.2614, 5.1932</td>
<td>9.8226, —</td>
<td></td>
</tr>
<tr>
<td>( (0.3, 0.3, 0.3) )</td>
<td>Average</td>
<td>29.2365, 28.7285, 28.3641, 29.3133</td>
<td>41.4440, —</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>1.6670, 2.6959, 3.7251, 6.6207</td>
<td>20.8639, —</td>
<td></td>
</tr>
<tr>
<td>Overall</td>
<td>Average</td>
<td>29.9856, 29.9655, 29.9879, 30.4684</td>
<td>32.0525, —</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>1.2100, 1.5418, 1.8460, 2.6979</td>
<td>5.3790, —</td>
<td></td>
</tr>
</tbody>
</table>

*The prior mean of \( N \) in the Bayesian models, except the one with the noninformative prior, is set equal to \( N = 30 \).

Figure 7. Nonconvergence rate of the nonlinear regression model in 10,000 simulation runs.

Figure 8. Mean absolute percentage error of various models in 10,000 simulation runs.

Often fails to converge. In practice, inspector competencies are often known in advance (e.g., experts versus trainees), and more competent inspectors are usually assigned first. Thus, it would be interesting to test the optimality of the “expert-first” sequencing rule in future study (Drury et al. 1986).

7. Concluding Remarks

For the problem of estimating the total number of defects in the sequential inspection plan, we developed the maximum likelihood method and the Bayesian method, and then compared their performances with that of the existing nonlinear regression method. We showed in the simulation study that the Bayesian method with noninformative priors performed very well, followed by the maximum likelihood method. Although we considered the case in which there are only three inspectors, we expect that the foregoing results will be similar for other cases with more than three inspectors. Thus, in a specific inspection environment, it would be worthwhile to use various estimation methods and compare their point estimates and measures of accuracy.

We believe that any prior knowledge can improve the performance of the Bayesian method even further. For example, we may first find the maximum likelihood estimates and then use the estimates in specifying the parameter values of the prior distributions in the Bayesian model, which is shown to be robust to miss-specified prior distributions. As in most cases of acceptance sampling plan, we believe that “our prior knowledge does not need to be very precise, provided that the form of the distribution chosen is reasonable” (Wetherill and Chiu 1975).

As shown in the simulation study, the existing nonlinear regression method has certain limitations. First, one
The limitation is that the parameter estimators and test statistics are derived from large-sample theory; their behavior in small samples is currently not known. When the total number of defects $N$ is small in a high-quality product and/or the inspector’s detection probability $p$ is relatively low, the inspection results $x = \{x_1, x_2, \ldots, x_k\}$ may contain zero values. As suggested by Bonett and Woodward (1994), these zero values may be arbitrarily replaced with
small nonzero values (e.g., 1/k), but its small-sample effect requires further investigation.

Second, another computational limitation is that the nonlinear regression algorithm does not always converge to stationary points. If the determinant of a matrix of regression parameters converges to zero, the inverse of the parameter matrix does not exist and the matrix-based nonlinear regression method fails to produce the estimates of N and p.

In our Monte Carlo simulations, we faced these limitations frequently when the detection probability is low and/or the number of defects in the data set is low. In such cases, the data set x = {x1, x2, ..., xN} could contain zero values and/or the elements x_i in the set are not decreasing as anticipated in practical situations. One of the modified methods suggested by a reviewer is adding a constant (e.g., 1/i) to each x_i and then subtracting the sum of the constants from the final estimate of N. In the preliminary study, the modified method did not improve the prediction accuracy, although it reduced the nonconvergence errors slightly. Another modification suggested by the same reviewer is that any nondecreasing elements x_i are replaced with the average of two adjacent elements; i.e., (x_i + x_{i+1})/2 if x_i < x_{i+1}. We found that the modified method reduces the nonconvergence errors dramatically. However, the nonlinear regression method—modified or not—clearly underperforms the maximum likelihood method and the Bayesian method with noninformative priors.

The sequential selection problem considered in this paper can be extended in many different directions, with varying degrees of difficulty. As we mentioned earlier, it would be interesting to consider the problem of selecting the best subset of inspectors with different competencies and wages, determining the order of inspections, and allocating the time for each inspector in an optimal way (Raz and Bricker 1993). It is also interesting to compare the performance of a sequential inspection plan with that of a parallel inspection plan. Furthermore, as pointed out by a reviewer, prior information could be used to modify the Bonett-Woodward estimates. In such a case, a comparison of the informative Bayesian models with the modified Bonett-Woodward estimates could be another interesting area for future research. We believe that the models developed in the paper would be good stepping stones for future research on various types of multiple inspection plans.

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