Test 3 2405 Name _______________

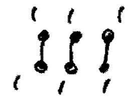
I. Graph Theory

1. (13 pts) True/False.

a) **True** or **False**: There exists a graph on 4 vertices and 6 edges. 

b) **True** or **False**: There exists a graph on 4 vertices and 8 edges. max edges 6

c) **True** or **False**: There exists a graph on 5 vertices with degree sequence 3, 3, 3, 2, 2. → A graph must have an even number of odd degrees

d) **True** or **False**: There exists a graph on 6 vertices with degree sequence 1, 1, 1, 1, 1, 1. 

e) **True** or **False**: If two graphs are isomorphic, then the two graphs have the same degree sequence.

f) **True** or **False**: If two graphs have the same degree sequence, then the two graphs are isomorphic. See problem #3

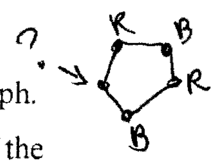
g) **True** or **False**: A cycle on n vertices has n edges (assuming n is at least 3).

h) **True** or **False** A path on n vertices has n *-1* edges. Examples 

i) **True** or **False**: A complete graph on n vertices has $n(n-1)$ edges. see 1 part a

j) **True** or **False**: The maximum degree of the complement of a cycle on 8 vertices is 6. degree is 5 in the complement 

k) **True** or **False**: The maximum degree of the complement of a path on 8 vertices is 6.

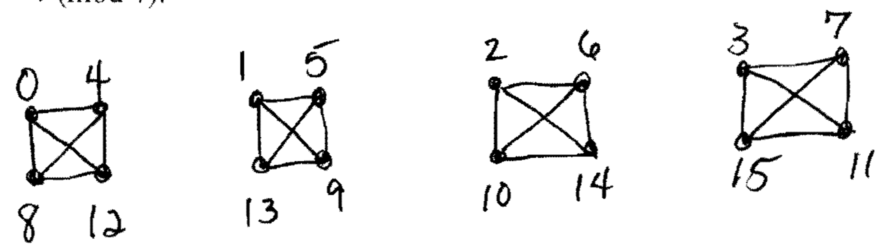
l) **True** or **False**: The cycle graph on 5 vertices is also a bipartite graph. ? 

m) **True** or **False**: For any graph G the number of edges is the sum of the degrees.

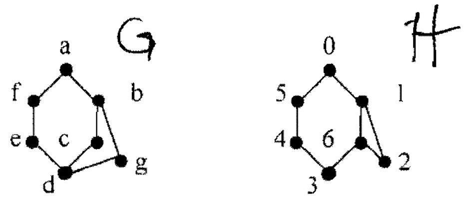
The H.S.T states that

$$\sum_{v \in V(G)} \deg v = 2 \cdot \text{number of edges}$$

2. (8 pts) Draw the graph defined by on vertex set {0, 1, 2, 3, 4 , 5 , 6, 7, 8, 9, 10, 11, 12, 13,14, 15} such that two vertices u and v are adjacent if and only if $u \equiv v \pmod{4}$.

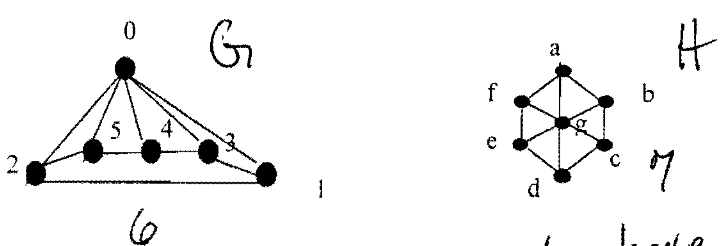


3. (6 pts) Prove or disprove that following two graphs are isomorphic.



An isomorphism (function) would need to map b to 1 and d to 6 or b to 6 and d to 1 because they are the only degree 3 vertices but in either case ~~can iso~~ the function could not be edge preserving since $1 \sim 6$ but $b \not\sim d$.

4. (6 pts) Prove or disprove that following two graphs are isomorphic.



The two graphs have different number of vertices, so NO function from G to H can be onto H, and NO function from H to G can be one-to-one. Thus $G \not\cong H$.

Def: $G \cong H \iff \exists f: V(G) \to V(H)$ such that
f is onto
f is one-to-one
f is edge preserving

Integers and Division

5. (8 pts)

i. Find GCD($2^4 \cdot 5^4 \cdot 7^3 \cdot 17,\ 2^4 \cdot 5^3 \cdot 7^2 \cdot 17^3 \cdot 19$)

$$\boxed{2^4 \cdot 5^3 \cdot 7^2 \cdot 17}$$

ii. Find LCM($2^4 \cdot 5^4 \cdot 7^3 \cdot 17,\ 2^4 \cdot 5^3 \cdot 7^2 \cdot 17^3 \cdot 19$)

$$\boxed{2^4 \cdot 5^4 \cdot 7^3 \cdot 17^3 \cdot 19}$$

iii. If the product of two integers $a \cdot b = 2^5 \cdot 3^4 \cdot 5^7 \cdot 7^3$ and the GCD of a and b is $2^2 \cdot 3 \cdot 5^2 \cdot 7$, then what is the LCM of a and b.

Since $a \cdot b = gcd(a,b)\, lcm(a,b)$

$$lcm = \frac{a \cdot b}{gcd(a,b)} = \frac{2^5 3^4 5^7 7^3}{2^2 \cdot 3 \cdot 5^2 \cdot 7} = \boxed{2^3 3^3 5^5 7^2}$$

6. (6 pts) *Demonstrate the Euclidean Algorithm* for finding the GCD of 2024 and 1024. (Note that the GCD is not enough, you must demonstrate the Euclidean algorithm).

$$2024 = 1 \cdot 1024 + 1000$$
$$1024 = 1 \cdot 1000 + 24$$
$$1000 = 41 \cdot 24 + 16$$
$$24 = 1 \cdot 16 + 8 \quad \text{gcd}$$
$$16 = 2 \cdot 8 + 0$$

7. (10 pts) True/False.

a. **True** or **False** : The integers have the closure property with respect to subtraction. $a, b \in \mathbb{Z} \to a - b \in \mathbb{Z}$

b. **True** or **False** : The sum of two consecutive integers is odd. $n + n+1 = 2n+1$

c. **True** or **False**: The product of two consecutive integers is odd. $n(n+1)$ not odd

d. **True** or **False**: 8 is a divisor of 64

e. **True** or **False** 64 is a factor of 8

f. **True** or **False**: For a and b nonzero integers, $(a \mid b \ \wedge \ b \mid c) \to a \mid c$

g. **True** or **False**: For a nonzero integer a, $a \mid (b + c) \to a \mid c$ $\quad 3 \mid (4+5)$ but $3 \nmid 4 \wedge 3 \nmid 5$

h. **True** or **False**: *12* mod *4 = 0* $\quad 12 \div 4$ has quotient 3 remainder 0

i. **True** or **False**: *-4≡12* mod *4* $\quad 4 \mid (-4-12)$

j. **True** or **False**: If $a \equiv b$ mod m and $c \equiv d$ mod m, then $a+b \equiv c+d$ mod m

$0 \equiv 4 \bmod 4 \qquad 5 \equiv 1 \bmod 4$

but $0+4 \not\equiv 5+1 \bmod 4$

8. (9 points) Store the following student ID numbers in memory locations 0-9 using a hashing function $f(s) = s \bmod 10$ (assume they are processed in the order given here). Explain how you managed collisions.

658779, 658772, 657102, 648559, 657991, 657122, 658881, 648557, 657100

Memory location	ID numbers
0	648559
1	657991
2	658772
3	657102
4	657122
5	658881
6	657100
7	648557
8	
9	658779

see classnotes or textbook for a reminder

9. (8 pts) Let a, b and c be integers such that c is nonzero. *Complete the proof of* if $c \mid a$ and $c \mid b$, then $c \mid (xa + yb)$ for any integers x and y.

Proof. Let a, b and c be integers such that c is nonzero. Assume that $c \mid a$ and $c \mid b$. Then by def of divisibility, $a = ck$ for some integer k and $b = cj$ for some integer j. Let x and y be any integers. Then by substitution,

$$\begin{aligned} xa + yb &= x(ck) + y(cj) \\ &= c(xk + yj) \quad \text{by Arithmetic.} \end{aligned}$$

Since x, k, y and j are integers and the integers have closure with respect to multiplication and addition, $xk+yj$ is an integer. Thus, by definition of divisibility, $c \mid (xa + yb)$. Hence, if $c \mid a$ and $c \mid b$, then $c \mid (xa + yb)$ for any integers x and y. □

Mathematical Induction

10. (10 pts) Use the principal of math induction to prove the following.

$$\forall n \in N, \sum_{i=0}^{n}(4i+1) = (n+1)(2n+1)$$

proof: Since $\sum_{i=0}^{0}(4i+1) = 4\cdot 0+1 = 1 = (0+1)(2\cdot 0+1)$,

our basis step is established.

Assume $\sum_{i=0}^{K}(4i+1) = (K+1)(2K+1)$ for some $K \in N$.

By Associativity,

$$\sum_{i=0}^{K+1}(4i+1) = \sum_{i=0}^{K}(4i+1) + 4(K+1)+1$$

$$= (K+1)(2K+1) + 4(K+1)+1 \quad \text{by inductive assumption}$$

$$\left.\begin{aligned} &= 2K^2+7K+6 \\ &= (2K+3)(K+2) \\ &= ((K+1)+1)(2(K+1)+1) \end{aligned}\right\} \text{arithmetic}$$

Thus by the PMI, $\forall n \in N \ \sum_{i=0}^{n}(4i+1) = (n+1)(2n+1)$ $\square$

Counting

11. (6 pts) For $A = \{1, 2, 3, 4, 5, 6, 7\}$, determine the following:

i. The **number** of all subsets of A.

2^7

ii. The **number** of functions from A to A?

7^7

iii. The **number** of one-to-one functions from A to A?

$7!$

12. (3 pts) Determine the **number** of bit strings of length *10* that contain at most *1* zero bit.
(a) *10* **(b)** 2^{10} **(c)** $2^{10}-1$ **(d)** *11* **(e)** none of these

0 1 1 1 1 1 1 1 1 1
1 0 1 - - - - - 1
1 1 0 . . . .
⋮
1 1 1 1 1 1 1 1 1 0
} 10 with 1 zero bit

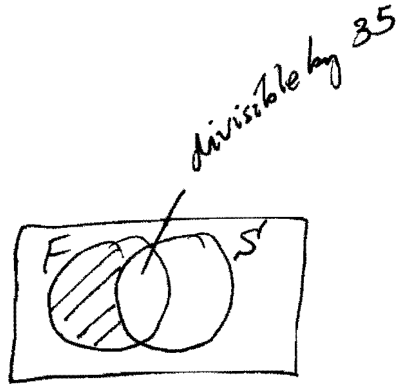
1 1 1 1 1 1 1 1 1 1 ← 1 with no zero bit

13. (6 pts) Find the number of positive integers less than or equal to 2000
a. that are divisible by 7.

$$\left\lfloor \frac{2000}{7} \right\rfloor$$

b. that are divisible by 5 but not 7.

$$\left\lfloor \frac{2000}{5} \right\rfloor - \left\lfloor \frac{2000}{35} \right\rfloor$$



14. (3 points) Among a group of 165 students, 80 are taking Cal I and 95 are taking CS I. How many are taking both of these courses?

Let A be students in CAL I and
B " " " CS I

Question $|A \cap B| = ?$

Since $|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cap B| = |A| + |B| - |A \cup B|$

$= 80 + 95 - 165$

$= \boxed{10}$