Permutations and Combinations

- What do we mean by a permutation of the elements of a set?

- There are a couple of ways to define this concept and we will look at two of them.

But first let us look at an example: Let us consider the elements *a*, *b*, and *c* as follows:

(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).

Each is a rearrangement of the three objects *a, b* and *c*.

Another way to look at each of the rearrangements is as a one-to-one function. For example:

	1	2	3
f_1			
f_2			
f_3			
f_4			
f_5			
f_6			

Question: How many rearrangements are there of an *n*-element set? (i.e. how many one-to-one functions are there from an *n*-element set to another *n*-element set?)

More generally, we are interested in the following types of permutations. Given *n* objects how many *r*-element rearrangements are there of these objects assuming that $n > r$.

Example: Given the *26* letters in the alphabet, how many *3*-letter arrangements are possible, assume no letter is repeated. Some examples of *3*-letter arrangements are *(a, b, c), (a, f, k), (p, b, n)* and so on.

Solution: ***Each*** *3*-letter arrangement is a one-to-one function. For example: *(a, f, k)* is the function $f : \{1, 2, 3\} \rightarrow \{a, b, c, \ldots, y, z\}$, define as $f(1) = a$, $f(2) = f$, and $f(3) = k$.

Thus there are _______________ *3*-letter arrangements of the alphabet.

Thus on the one hand: An **k-permutation** of an *m*-element set with distinct elements, *S*, is a one-to-one function $f : \{1, 2, \ldots, k\} \rightarrow S$.

Another definition: A **k-permutation** of an *m*-element set with distinct elements is an ordered arrangement of *k* elements of the set.

Theorem. The number of *k*-permutations of a set with *m* distinct elements, where $0 \le k \le m$, is

$$P(m,k) = (m)_k = \frac{m!}{(m-k)!}$$

Proof. Each *k*-permutation is a one-to-one function from a *k*-element set to an *m*-element set. Thus by Example from previous note (discussed in class), the result follows.

Def. A **k-combination** of elements of an *m*-element set is an unordered selection of *k* elements from the set, i.e. an *r*-combination is simply a subset of the set with *r* elements.

The number of *k*-permutations of a set with *m* distinct elements is **not equal** to the number of *k*-combination of elements of an *m*-element set, but they are related.

Theorem. The number of *k*-combinations of a set with *m* distinct elements, where $0 \le k \le m$, is

$$C(m,k) = \binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

Proof: Let $C(m,k) = n$. The *k*-permutations of a set with *m* distinct elements can be obtained by forming the *n* *r*-combinations of the set, and then ordering the elements in each *r*-combinations, which can be done in $P(k,k)$ ways. Consequently,

$$P(m,k) = C(m,k)\, P(k,k).$$

This implies that

$$C(m,k) = \frac{P(m,k)}{P(k,k)} = \frac{m!/(m-k)!}{k!/(k-k)!} = \frac{m!}{k!(m-k)!}.$$

Examples:

i) How many subsets of the alphabet are there?

ii) How many subsets of the alphabet contain exactly *3* letters?

iii) How many subsets of the alphabet contain exactly *5* letters?

iv) How many subsets of the alphabet contain at most *5* letters?

v) How many subsets of the alphabet contain at more than *5* letters?

Example: In how many ways can two students be chosen from a class of *18*?

Examples:

i) how many permutations of the letters abcdefgh are there?
ii) how many permutations of the letters abcdefgh contain the string dfg?

Example: In how many ways can two students be chosen from a class of *18* if one of them receives an *A* and the other receives a *B*?

Example: How many ways are there to select 5 players from a 10-member tennis team to make a trip to another school?

Example: How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

Corollary: Let *m* and *k* be nonnegative integers with $k \le m$.

Then $\binom{m}{k} = \binom{m}{m-k}$.

Proof: