Mathematical Induction

Theorem. (The Principle of Math Induction) (PMI)
Let *P(n)* be a proposition.
If (i) *P(1)* is true
(ii) If *P(k)* → *P(k+1)* for every positive in integer
then *P(n)* is true for every positive integer.
Note we will skip this proof and practice using it instead.

How to use the Principle of Mathematical Induction:

Step 1: Identify the math statement to be proven.
Step 2: Show that the statement is true for the natural number 1.
Step 3: Show that if we assume that the statement is true for some *k*, then it follows that the statement must also be true for *k+1*, i.e. property (ii).
Step 4: Conclusion: By the Principle of Math Induction….

Exercise 1: Prove the following using mathematical induction

Theorem $\forall n \in N, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$**.**

Proof: **Step 1:** Let ***P(n)*** be the statement $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Step 2:

Example 2: Prove that for all $n \in N$, $\left(1+\frac{1}{2}\right)^n \geq 1 + n/2$.

Proof: Let *P(n)* ____.

Since $\left(1+\frac{1}{2}\right)^1 = 1 + 1/2$, *P(1)* is true.

Assume __________________________. This means that

____.

$$\left(1+\frac{1}{2}\right)^{k+1} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$\geq \left(1+\frac{1}{2}\right)\left(1+\frac{k}{2}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$= 1 + \frac{1}{2} + \frac{k}{2} + \frac{k}{4} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$= 1 + \frac{k+1}{2} + \frac{k}{4} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$\geq 1 + \frac{k+1}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Thus if *P(n)* is true, then *P(n+1)* is also true. Hence by PMI,

$\left(1+\frac{1}{2}\right)^n \geq 1 + n/2$ ______________. QED

Example 3: Can PMI be used to show that $\forall n \in N, \ n = n + 1$?

Solution: Let *P(n)* be the statement *n = n +1*. Assume *P(k)* is true, that is assume *k = k +1* for some integer *k*.

k + 1 = (k+1) + 1 since *P(k)* is true

= k + 2.

Thus *P(k+1)* is true whenever *P(k)* is true. Hence by PMI,….?

What happened? How could we prove this nonsense?

Example 4: Sums of Geometric Progressions. Use mathematical induction to prove the following formula for the sum of a finite number of terms of a geometric progression.

$$\sum_{j=0}^{n} ar^{j} = \frac{ar^{n+1} - a}{r - 1}, \qquad \text{when } r \text{ is not equal to } 1.$$

Example 5: Use mathematical induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$.

Example 6: Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

Example 7: Use mathematical induction to prove that if S is an n element set, then $|P(S)| = 2^n$ for every positive integer n.