Ch 2.1 Sets

Terms: The **objects** in a set are called the **elements**, or **members**, of the set. A set is said to contain its elements.

Notation: We use upper case letters to denote sets, e.g. *A, B, C, X,…*

Example: For the set A = {a, b, 3, 5}. The members of this set are _______.

Def: **Two sets are equal** if and only if they have the same elements.
Examples: Let $A = \{1, 2, 5\}$, $B = \{1, 2, 3\}$, and $C = \{5, 2, 1\}$. Sets A and C are equal (note the order does not matter), but no other combinations in this example are equal.

Notation/Definitions:

1. $\in$ is read as "**is an element of**".
   Example: Let A = {1, 2, 5}. $1 \in A$, $2 \in A$, but $3 \notin A$.

2. If a set is finite or has a pattern then the set can be described by listing the elements. But a more general way to describe a set is by the use of **set builder notation**.

   Examples:
   i. $R = \{x \mid x \text{ is a real number }\}$
      (The braces indicate a set and the vertical bar is read as "such that")
      This is read as " the set of all x such that x is a real number "

   ii. $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$
      This is read as " the set of all x such that x is an odd positive integer less than *10* ".
      (Note this set could have been listed O = { 1, 3, 5, 7, 9 }.)

3. The **universal set**, which we will denote as U, is the set of all objects under consideration in a given problem.

4. **A set A is said to be a subset of set B** if and only if every element of A is also an element of B.

In notation this is: $A \subseteq B \leftrightarrow \forall x(x \in A \to x \in B)$.
A is **not a subset** of *B* in notation: $A \not\subset B \leftrightarrow \exists x(x \in A \wedge x \notin B)$
Proof:

$A \not\subset B \Leftrightarrow$	
$\neg \forall x(x \in A \to x \in B) \Leftrightarrow$	By the negation of the definition of subset
$\exists x \neg (x \in A \to x \in B) \Leftrightarrow$	By the negation of the universal quantifier.
$\exists x \neg (x \notin A \vee x \in B) \Leftrightarrow$	Since $(p \to q) \Leftrightarrow (\neg p \vee q)$
$\exists x(x \in A \wedge x \notin B)$	??

End of Proof.

Examples: Let $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, $C = \{b, c, d\}$, and $D = \{c, b, a\}$. **Fill in the blank with the subset symbol or the not a subset of symbol.**

A		B
C		B
A		C
D		B
D		A

5. $\varnothing$ represents the **empty set** or **null set**, which is defined as the set with no elements.
Fact: Let the universal set be the collection of all sets.
$$\forall A,\ \varnothing \subseteq A.$$
Proof:

6. **Corollary**. A nonempty set has at least two subsets.

7. Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a **finite set** and n is the **cardinality** of S. The cardinality of S is denoted by $|S|$. A set is said to be **infinite** if it is not finite.
   Examples: 1. Let $A = \{1, 2, 5\}$. Then $|A| = 3$.
   2. Let S be the set of letters in the English alphabet. Then $|S| = 26$.
   3. $|\varnothing| =$ ________ 4. $|\{\varnothing\}| =$ _________
   5. $|\{1000\}| =$ _______ 6. $|\{\infty\}| =$ ________
   7. $|\{1,2,\ldots,1000\}| =$ _______ 8. $|\{1,2,1000\}| =$ ________

Definition: Given a set S, the **power set of S** is the set of all subsets of the set S. The power set of S is denoted by $\boldsymbol{P(S)}$.

Fact to be proven later: If $|A| = n$, then $|P(A)| = 2^n$.

Examples:

1. Let $A = \{a, b, c\}$. Describe $P(A)$.
   Solution: $P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
   Note just as describe in the fact above this power set has $2^3 = 8$ elements.
2. **You try**. Let $B = \{0, 1\}$. Describe $P(B)$.

3. $P(\varnothing) = ?$

4. Let A be any set. True/False.

1.	$\varnothing \subseteq A$	
2.	$\{\varnothing\} \subseteq A$	
3.	$\varnothing \subseteq P(A)$	
4.	$\{\varnothing\} \subseteq P(A)$	
5.	$\varnothing \in A$	
6.	$\{a\} \in A$	
7.	$\{a\} \subseteq A$	
8.	$\{a\} \in P(A)$	
9.	$\{a\} \subseteq P(A)$	
10	$\{\{a\}\} \subseteq P(A)$	

Definition: The **ordered n-tuple** $(a_1, a_2, a_3, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its n^{th} element.

Definition: $(a_1, a_2, a_3, \ldots, a_n) = (b_1, b_2, b_3, \ldots, b_n)$ if and only if
$(a_1 = b_1) \wedge (a_2 = b_2) \wedge \ldots. \wedge (a_n = b_n)$

Terminology: (a_1, a_2) is called an **ordered pair**.
(a_1, a_2, a_3) is called an **ordered triple**.

Definition: Let A and B be sets. The **Cartesian Product of A and B**, denoted $A \times B$, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$. Hence in set builder notation, the cartesian product is

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}.$$

Example: Let $A = \{a, b\}$ and $B = \{0, 1, 2\}$. Find $A \times B$ and $B \times A$.

Def: **Generalized Cartesian Product**: Let $A_1, A_2, \ldots, A_n$ be sets.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid (a_1 \in A_1) \wedge (a_2 \in A_2) \wedge \ldots \wedge (a_n \in A_n)\}$$