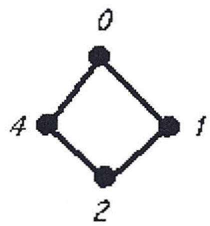
Ch 9.2 continued

Definition. Let G be a graph on vertex set V and edge set E. A subset of the vertices, $S \subseteq V$, is called ***independent*** if no two of vertices of S are adjacent.

Definition. A graph G is a ***bipartite graph*** if the vertices can be partitioned into two sets X and Y such that vertices of X determine an independent set, the vertices of Y too determine an independent set and $X \cap Y = \emptyset$ and $X \cup Y = V(G)$. A ***complete bipartite graph*** is a bipartite graph in which every edge of X is adjacent to every vertex of Y. We denote a complete bipartite graph as $K_{n,m}$, where one partition has n vertices and the other has m vertices.

Example : The graph C_4 is a bipartite graph since the vertex set can be partitioned as $X = \{0, 2\}$ and $Y = \{1, 3\}$, and the sets each are independent.



Exercise : Provide an explanation for the fact that the graph C_3 is a not a bipartite graph.

Exercise : Draw an example of a bipartite graph in which each of the parts has 4 vertices (that is the graph has 8 vertices) and the number of edges is 10.

Theorem. The Handshaking theorem. Let $G = (V, E)$ be an undirected graph with e edges. Then the sum of the degrees is equal to twice the number of edges.

Theorem. Let $G = (V, E)$ be an undirected graph has an even number of vertices of odd degree.

Exercise : How many edges are there in k-regular graph on n vertices?

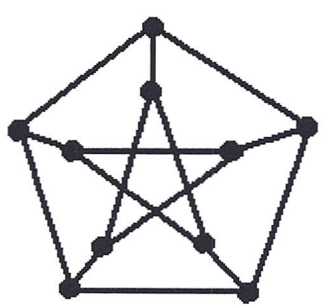
Exercise : Describe a formula for the average degree of a graph G in terms of the number of edges and the number of vertices?

Exercise If possible, construct a graph 3-regular graph on 5 vertices.

Exercise : If G is a 5-regular graph on n vertices, what can you deduce about n?

Definitions. A **subgraph of a graph G** is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; If H is a subgraph of G, this is denoted by $H \subseteq G$. This notation is in some ways unfortunate since of course H and G are not sets, such is the situation. A subgraph H of G in which every edge of G determined by the vertices of H is in $E(H)$ is called an **induced subgraph of G.**

Examples : (i) Show that C_5 is an induced subgraph of the Peterson graph.
(ii) Verify that the Peterson graph has twelve 5-cycles.



The Pigeonhole Principle

The pigeonhole principle is a useful tool that leads simpler elegant proofs. One version of it essentially states that any set of numbers has a number at least as large as the average.

Theorem (Pigeonhole Principle)
If a set consisting of more than kn objects is partitioned into n classes, then some class receives more than k objects.
Proof. We will use this fact without proof (see proof in textbook).

Proposition Every graph with at least two vertices has two vertices of equal degree.

Proof:

Ch. 9.3 Graph Isomorphisms

You should have noticed thus far that how we draw a graph does not alter what we may call the graph. Recall our example about the graphs we call paths you showed that the graph below on the left is a path on 6 vertices. But clearly the graph below on the right is also a path on 6 vertices. Thus in a very strong sense the graphs are the same. However, we do not say that the graphs are equal. The correct term for this idea is that the graphs are *isomorphic*. We investigate this concept and hope to alert you to some common mistakes on determining if graphs are isomorphic.

Figure 1.5. Two drawings of a path on 6 vertices.

Intuitive Definition: Two graphs G and H are isomorphic, denoted $G \simeq H$, if the graph H can be drawn on top of G so that the vertices of H and G match up and the edges of each graph also match up.

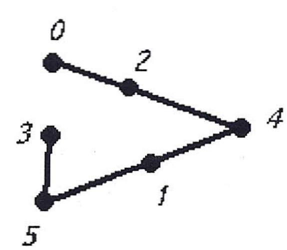
The intuitive defintion of isomorphic graphs should coincide with our previous discussion of the fact that the graphs in Figure 1.5 are essentially the same.

Formal Definition: Two graphs ***G* and *H* are isomorphic**, denoted $G \simeq H$, if there exists a one-to-one and onto function $f : V(G) \to V(H)$ with the property that $u \sim_G v$ if and only if $f(u) \sim_H f(v)$. (That is two vertices in $V(G)$ are adjacent if and only if there images under f are adjacent.) This property is called the **edge-preserving property**. If such

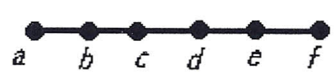
an

edge-preserving one-to-one and onto function exists it is called an **isomorphism**.

Exercise 2: In this example we practice exhibiting an isomorphism between the following two graphs. We have discussed these graphs repeatedly, and now it is time to "formally" show that the graphs are isomorphic.
Let G be the graph on vertex set $\{0, 1, 2, 3, 4, 5\}$ whose edge set is described by the diagram:



Let H be the graph on vertex set $\{a, b, c, d, e, f\}$ with whose edge set is described by the diagram:



Hint to Solution of Exercise :

Step 1: Once your intuition tells you that the graphs are almost certainly isomorphic, you must describe an isomorphism. In this particular example, you must describe a one-to-one and onto function from the set $\{0, 1, 2, 3, 4, 5\}$ to the set $\{a, b, c, d, e, f\}$.

Step 2: You must prove (in this simple case we will simply verify by hand) that the function you described has the edge preserving property.

Observation 1: That the function must be one-to-one and onto assures you that the vertices can be matched up. Note that for $f : V(G) \rightarrow V(H)$, if the graph G has fewer vertices, the function can not be onto (and one-to-one) $V(G)$. Further a function $f : V(G) \rightarrow V(H)$ in which H has fewer vertices cannot be one-to-one.

Observation 2: That the one-to-one and onto function is edge preserving assures you that the edges can be matched up.

Avoiding common errors for determining if graphs are isomorphic:

Example 3: The following two graphs are not isomorphic:

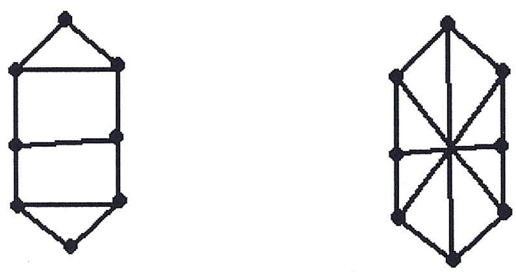


Figure 1.6 Graph H is on the left, and graph G is on the right.

Common Error 1: Students often believe that it is enough for two graphs to be isomorphic if they have the following common properties,

- Same number of vertices
- Each vertex has the same degree.

BUT these common properties are NOT enough to guarantee that two graphs are isomorphic. The graphs in Example 1.19 have the same number of vertices, and each vertex has the same degree, namely degree 3. But the graphs in the example are not isomorphic since the edges of neither of the 3-vertex cliques in H can ever be preserved by any one-to-one and onto function. A more formal argument that the graphs are not isomorphic is that one graph has clique number 2, while the other has clique number 3.

Fact: In general $G \simeq H$ if and only if aside from the labeling of vertices the graphs have identical graph invariants.

Common Error 2: Students often believe that the they can show that two graphs are isomorphic using the fact above, BUT it is impossible to imagine (much less examine) all possible graph invariants.

There is no short cut for showing that two graphs are isomorphic, one must define the edge-preserving one-to-one and onto function (i.e. the isomorphism.)

Summary:

1. To show that two graphs are isomorphic you must describe the one-to-one and onto function, and verify that it is edge-preserving.
2. However, to show that two graphs are not isomorphic it is enough to point out some property or invariant in which they differ.

Exercise 4: Determine if the following graphs are isomorphic