

## Ch 2.2 Set Operations

**Definitions.** Let  $A$  and  $B$  be sets.

1. The **union of sets A and B** is denoted and defined as follows:  

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$
2. Fact:  $|A \cup B| = |A| + |B| - |A \cap B|$
3. The **intersection of sets A and B** is denoted and defined as follows:  $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$
4. Two sets  $A$  and  $B$  are said to be **disjoint** if  $A \cap B = \emptyset$ .
5. The **symmetric difference of two sets A and B**, denoted by  $A - B$ , is defined as follows:  $A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$ .
6. The **complement of a set A**, denoted by  $\bar{A}$ , is defined as follows:  

$$\bar{A} = \{x \mid x \notin A\}.$$

**Example 1:** Prove  $(A - B) - C \subseteq A - C$ .

**Example 2:** Prove DeMorgan's Law:  $\overline{A \cup B} = \bar{A} \cap \bar{B}$ .

Recall that  $X = Y$  if and only if  $X \subseteq Y$  and  $Y \subseteq X$ .

Proof:

**Example 3:** Prove  $A \cup U = U$  (Note that this is one of the domination laws)

**Example 4:** Prove DeMorgan's Law:  $\overline{A \cap B} = \bar{A} \cup \bar{B}$ .

**Example 5:** Find the Error in the following proof. Note the statement is true but the proof has errors.

Prove that  $A - B = A \cap \bar{B}$

Proof.

$A - B$	$= \{x \mid (x \in A) \cap (x \notin B)\}$	By definition of set difference
	$= x \mid (x \in A) \cap (x \in \bar{B})$	By definition of the complement of a set
	$= x \mid A \cap \bar{B}$	By definition of the intersection

End of Proof.

**Set Identities:** Table 1 on page 49 of our text consists of set identities, many of which we will prove.

Set Identities		
<i>Identity</i>	<i>Name</i>	
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws	
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws	
$A \cup A = A$ $A \cap A = A$	Idempotent laws	
$\overline{\overline{A}} = A$	Complementation law	
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws	
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws	
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws	
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws	

There are many proof techniques used to prove set identities (we will omit membership tables.) Two of these methods were illustrated above and below is a demonstration of another method.

**Example 6:** Show  $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Proof.

$\overline{A \cup (B \cap C)}$	$= \overline{A} \cap \overline{(B \cap C)}$	By DeMorgan's Law for sets.
	$= \overline{A} \cap (\overline{B} \cup \overline{C})$	By DeMorgan's Law for sets.
	$= (\overline{B} \cup \overline{C}) \cap \overline{A}$	
	$= (\overline{C} \cup \overline{B}) \cap \overline{A}$	

End of Proof.

## Generalized Unions and Intersections

Let  $A_1, A_2, \dots, A_n$  be sets.

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n.$$

**Note:** an element is in the union if it is in at least one of the  $A_i$  for  $i=1, 2, 3, \dots, n$ .

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n.$$

**Note:** an element is in the intersection if it is in each of the  $A_i$  for  $i=1, 2, 3, \dots, n$ .

### Example 7.

Let  $A_i = \{1, 2, 3, \dots, i\}$  for  $i = 1, 2, 3, \dots$

- (a) List the elements of  $A_1 =$
- (b) List the elements of  $A_2 =$
- (c) List the elements of  $\bigcup_{i=1}^5 A_i =$
- (d) If  $n$  is greater than 2, then list the elements of  $\bigcup_{i=1}^n A_i =$
- (e) List the elements of  $\bigcap_{i=1}^5 A_i =$
- (f) If  $n$  is greater than 2, then list the elements of  $\bigcap_{i=1}^n A_i =$

### Example 8.

Let  $A_i = \{i, i+1, i+2, \dots\}$  for  $i = 1, 2, 3, \dots$

- (a) If  $n$  is greater than 1, find  $\bigcup_{i=2}^n A_i =$
- (b) If  $n$  is greater than 1, find  $\bigcap_{i=2}^n A_i =$