Ch. 1.6 Introduction to Proofs

The following techniques for methods of proofs are discussed in our text:

- Vacuous proof
- Trivial proof

- Direct proof
- Indirect proof (our book calls this by contraposition)
- Proof by contradiction
- Proof by cases
- (later) mathematical induction

- A **vacuous proof** of an implication happens when the hypothesis of the implication is always false.

Example 1: (Vacuous proof) Prove that if x is a positive integer and $x = -x$, then $x^2 = x$.
proof.

- An implication is **trivially true** when its conclusion is always true.

- A declared mathematical proposition whose truth-value is unknown is called a **conjecture**.

One of the main functions of a mathematician (and a computer scientist) is to decide the truth-value of their claims (or someone else's claims).

- If a conjecture is proven true we call is a *theorem*, *lemma* or *corollary*; if it proven false, then usually discarded.

- A **proof** is a sequence of statements bound together by the rules of logic, definitions, previously proven theorems, simple algebra and axioms.

Definition: An integer n is **even** if there exists an integer k such that $n = 2k$.
An integer n is **odd** if there exists an integer k such that $n = 2k + 1$.
Exercise: Give and example of an integer that satisfies the definition and of one that does not satisfy the definition.

Axiom (Closure of addition over the integers): If a and b are integers, then $a + b$ is an integer.

Axiom (Closure of multiplication over the integers): If a and b are integers, then $a \cdot b$ is an integer.

Example 2: Provide a counterexample to the statement that the product of two irrational numbers is irrational.

Lemma 1. If n is even, then n^2 is even.
Proof:

Lemma 2. If n^2 is even, then n is even.
Proof:

Theorem 1: An integer n is even if and only if n^2 is even.
Proof: If n is even, then n^2 is even is true by Lemma 1. The converse, if n^2 is even, then n is even is true by Lemma 2. Hence the biconditional statement n is even if and only if n^2 is even is true. □

Example 5: Prove that the sum of two odd integers is even. i.e. If p and q are odd integers, then $p + q$ is an even integer.
Proof:

Summary of the form of 2 techniques. If we are proving $p \rightarrow q$, then

A **direct proof** begins by assuming *p* is true. : : until we conclude *q*.	An **indirect proof** begins by assuming ~*q* is true. : : until we conclude ~*p*.

Example 6: Prove that $\sqrt{2}$ is irrational.

Proof: Assume by way of contradiction that $\sqrt{2}$ can be represented as a quotient of two integers *p/q* where *q* is not zero. Further, we assume that *p/q* is in lowest terms, i.e. we assume that

the integers *p* and *q* have no common factor. (1)

Thus, by assumption $\sqrt{2} = p/q$, and now squaring both sides yields

$$2 = \frac{p^2}{q^2} \qquad \text{or} \qquad p^2 = 2q^2 \qquad (2)$$

This implies that p^2 is even, and by Theorem 1, *p* must also be even. So $p = 2k$ for some integer *k* and substitute into the second equation of (2). Now by cancellation we see that

$$q^2 = 2k^2. \qquad (3)$$

This says that q^2 is even, and again by Theorem 1, *q* must also be even. From statements (2) and (3), it follows that

p and *q* both have *2* as a common factor. (4)

Statements (1) and (4) are contradictory. Thus, $\sqrt{2}$ is not rational.

□

Summary. If we are proving $p \rightarrow q$, then

A **direct proof** begins by assuming *p* is true. : : until we conclude *q*.	An **indirect proof** begins by assuming ~*q* is true. : : until we conclude ~*p*.	An **proof by contradiction** begins by assuming $p \wedge \sim q$ is true. : : until we reach a contradiction

Example 7: Prove that if $3n + 2$ is odd, then n is odd.

i. Write the proposition in symbolic logic notation.

ii. Write the negation of the proposition in symbolic logic notation.

iii. Give a proof by contradiction

Proof:

Example 8: Prove that there is an even prime number.

Definition. Let x be a real number. Then $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$.

Example 9: Prove if x is a real number, then $|-x| = |x|$.

Definition. A function f:$A \rightarrow$B is *one-to-one* if and only if

$$\forall x \forall y\,(\,f(x) = f(y) \rightarrow x = y\,),$$

which is logically equivalent to its contrapositive

$$\forall x \forall y\,(x \neq y \rightarrow f(x) \neq f(y)).$$

Example 10: Prove that the real valued function $f(x) = x + 1$ is one-to-one.

Example 11: Prove the following statements about an integer x are equivalent.

(i) $3x+2$ is even
(ii) $x+5$ is odd
(iii) x^2 is even