Predicates and Quantifiers

Recall that the expression $x > 3$ is not a proposition. Why?

***P(x)* Notation**:

- We will use the propositional function notation $P(x)$ to denote the expression " *x has property P*".
- As soon as x is assume to be some value then $P(x)$ has a truth value.

The goal of the section is to quantify such statements so that the result is a proposition. The process is very much in the spirit of the next example.

Example: Let $P(x)$ denote the statement "$x > 3$".
What are the truth values:
i. $P(4)$ True or False ?
ii. $P(2)$ True or False ?
iii. $P(y)$ True or False ?

Example: Let $Q(x,y)$ denote the statement "$x = y + 3$".
What are the truth values:
i. $Q(7,4)$ True or False ?
ii. $Q(2,2)$ True or False ?

Quantifiers

Def. The **universe of discourse** for a math statement is the domain of that statement.

Def. The **universal quantification of *P(x)*** is the proposition
" *P(x) is true for all values of x in the universe of discourse*."

- This proposition is denoted by $\forall x\ P(x)$.
- The proposition $\forall x\, P(x)$ is read as "for all x $P(x)$" or "for every x $P(x)$".
- The symbol $\forall$ is called the **universal quantifier**.

Def. The **existential quantification of *P(x)*** is the proposition
" *There exists an element x in the universe of discourse for which P(x) is true*."

- This proposition is denoted by $\exists x\, P(x)$.
- The symbol $\exists$ is called the **existential quantifier**.
- The proposition $\exists x\, P(x)$ is read as "for some x $P(x)$" or "there exists an x such that $P(x)$".

The Truth Values of $\forall x\, P(x)$ and $\exists x\, P(x)$

Example 1: Suppose that the universe of discourse for x is the set of nonnegative integers {0, 1, 2, 3, 4, 5, …}

What is the truth value of $\forall x\ (x > 0)$? ________

What is the truth value of $\exists x\ (x > 0)$? ________

Example 2: Suppose that the universe of discourse for $P(x)$ is the small set $\{ x_1, x_2, x_3, x_4 \}$.

Since

$$\forall x\, P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge P(x_4)$$

when is $\forall x\, P(x)$ true? ___________________________________

when is $\forall x\, P(x)$ false? ___________________________________

Since

$$\exists x\, P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee P(x_4)$$

when is $\exists x\, P(x)$ true? ___________________________________

when is $\exists x\, P(x)$ false? ___________________________________

Note: The above observations also hold for infinite universe of discourse sets.

Examples 3:

Where $P(x)$ denotes:	Universe of Discourse for $P(x)$	Truth value of $\exists x\, P(x)$	Truth value of $\forall x\, P(x)$
$x = x + 3$	Real numbers		
$x = x + 0$	Real numbers		
$0/x = 0$	Integers		
$0 - x = x$	Positive Integers		
$0 - x = x$	Natural numbers		

Example 4:

(i) What is the truth value of $[\forall x P(x) \rightarrow \exists x P(x)]$? _____________

(ii) What is the truth value of $[\exists x P(x) \rightarrow \forall x P(x)]$? _____________

Example 5: Let $P(x,y)$ denote the statement "$x^2 = y$". Suppose that the universe of discourse is the set of real numbers.

- Which of the following is the English equivalent to $\exists y\ \forall x\ P(x,y)$

(a) For every real number y there is a real number x such that $x^2 = y$.
(b) For every real number y and for every real number x it is the case that $x^2 = y$.
(c) There is a real number y such that for every real number x it is the case that $x^2 = y$.

- What is the truth value of $\exists y\ \forall x\ P(x,y)$?

Example 6: Let $P(x,y)$ denote the statement " $x^2 = y$ ". Suppose that the universe of discourse is the set of real numbers.

- Which of the following is the English equivalent to $\forall y\ \exists x\ P(x,y)$?

(a) For every real number y there is a real number x such that $x^2 = y$.
(b) For every real number y and for every real number x it is the case that $x^2 = y$.
(c) There is a real number y such that for every real number x it is the case that $x^2 = y$.

- What is the truth value of $\forall y\ \exists x\ P(x,y)$?

Definition. A real number x is a **rational number** if can be expressed as the ratio of two integers a and b with b not zero, that is if $x = \frac{a}{b}$, with $b \neq 0$.

Example 7: Let $Q(x)$ denote the statement " x is a rational number ", and $I(y)$ denote the statement " y is an integer. Assume the universe of discourse is the set of real numbers.

i. $\exists y\ (Q(y) \wedge I(y))$

- Which of the following is the English translation?

(a) There is a rational number y which is a rational number and also an integer.
(b) There is a real number y which is a rational number and also an integer.
(c) Every real number y is a rational and also an integer.

- What is the truth value? ____________

ii. $\forall y\ (Q(y) \wedge I(y))$

- Which of the following is the English translation?

(a) There is a rational number y which is a rational number and also an integer.
(b) There is a real number y which is a rational number and also an integer.
(c) Every real number y is a rational or it is an integer.
(d) Every real number y is a rational and also an integer.

- What is the truth value? ___________

iii. $\forall y\ (I(y) \rightarrow Q(y))$

- Which of the following is the English translation?

(a) Every real number y is an integer and also a rational number.
(b) There is a real number y such that if y is an integer then y is a rational number.
(c) For any real number y, if y is an integer then y is a rational number.

- What is the truth value? ___________

iv. $\forall y\ (\neg I(y) \rightarrow Q(y))$

- What is the English translation? ___________________________
- What is the truth value? ___________

v. $\forall y\ (Q(y) \rightarrow I(y))$

- What is the English translation? ___________________________
- What is the truth value? ___________

Example 8: Assume the universe of discourse is the set of all students at UHD. Let $C(x)$ be " x has a computer" and let $F(x,y)$ be " x and y are friends."

i. Translate $C(Judy)$ into English. _______________________

ii. Which is the English translation of $\exists y(C(y) \wedge F(Judy, y))$.

(a) Judy has a friend who has a computer.
(b) Judy has a computer and a friend.
(c) There is a student who has a friend and a computer.

iii. Translate $C(Judy) \wedge \exists y(C(y) \wedge F(Judy, y))$ into English.

____

iv. Translate $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y))$ into English.

____

v. Which of the symbolic logic propositions is equivalent to the sentence " *There is a student at UHD who is friends with every UHD student*."

(a) $\exists x\, C(x)$ (c) $\exists y \forall x\, F(x,y)$

(b) $\exists y\, F(x,y)$ (b) $\forall y \exists x\, F(x,y)$

vi. Which of the symbolic logic propositions is equivalent to the sentence " *There is a student at UHD who is friends with only one other UHD student*."

(a) $\exists x \exists y\, [(x \neq y) \wedge F(x,y)]$

(b) $\exists x \exists y \forall z\, [(x \neq y) \wedge F(x,y)] \wedge [(y \neq z) \rightarrow \neg F(x,z)]$

(c) $\exists x \forall y \forall z\, [(x \neq y) \wedge F(x,y)] \wedge [(y \neq z) \rightarrow \neg F(x,z)]$

vii. Which of the symbolic logic propositions is equivalent to the sentence " *If a UHD student has a friend then the friend has a computer*."

(a) $\exists x \exists y\, [F(x,y) \rightarrow C(y)]$

(b) $\forall x \exists y\, [F(x,y) \rightarrow C(y)]$

(c) $\exists x \forall y\, [F(x,y) \rightarrow C(y)]$

viii. Express the sentence in terms of *C(x), F(x,y)*, logical connectives and quantifiers.

" *There is a student at UHD who is friends with every UHD student who has a computer*."

____

Example 9: Assume the universe of discourse is the set of elements a set *U*. Let *A* and *B* be subsets of *U*. A function from *A* to *B* is *one-to-one* if for every *a* and *b* in *A*, *a=b* whenever *f(a) = f(b)*.

i) Express the definition in terms of logical connectives and quantifiers.

____

ii) Using part i) explain when a function is not *one-to-one*.

____

Negations of Quantified Expressions

Example 10: Consider the proposition " *Every student in the class has taken a course in calculus*."

- When is this proposition true? _________________

- Write the proposition in symbolic logic notation.
  ____
  ____

- Which of the following is the negation of "*Every student in the class has taken a course in calculus*."
  (a) No student in the class has taken calculus.
  (b) Some students in the class have not taken calculus
  (c) Some student in the class has not taken calculus

- Write the negation of the proposition in symbolic logic notation.
  ____
  ____

- When is the negation of the proposition "*Every student in the class has taken a course in calculus*" true? _______________________

Fact:

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$
$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

Example 11:

(iii) What is the truth value of $[\forall x P(x)] \vee [\neg \forall x P(x)]$? _____________

(iv) What is the truth value of $[\forall x P(x)] \vee [\exists x \neg P(x)]$? _____________

(v) What is the truth value of $[\forall x P(x)] \wedge [\exists x \neg P(x)]$? _____________

Example 12: Write a statement equivalent to $\neg \forall y \exists x (P(x, y) \vee Q(x, y))$ so that the negation appears only within predicates (i.e. not in front of quantifiers.)