Ch. 2.4: Sequences and Summations

Def. Recall the natural numbers are denoted $N = \{1, 2, 3, \ldots\}$ and the whole numbers $W = \{0, 1, 2,, 3, \ldots\}$. Let S be a subset of integers. A **sequence** is a function $f{:}N \to S$ or $f{:}W \to S$. Instead of $f(n)$ we use the notation a_n to denote the image of the integer n. We call a_n the n[th] term of the sequence.

Sequences and summations occur in many Math and CS areas of study for example Linear algebra, Calculus, Number Theory, Complexity of Algorithms and Discrete Structures.

Notation: We use the notation $\{a_n\}$ to describe the sequence $a_1, a_2, a_3, \ldots$.

Example: Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$ (*the general formula of the sequence.*)
The list of the terms of this sequence, beginning with a_1, namely,

$a_1, a_2, a_3, a_4, \ldots$ starts with $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Example: Consider the sequence $\{b_n\}$, where $b_n = (-1)^n$ (*also called the rule of the sequence.*)
The list of the terms of this sequence, beginning with b_1, namely,

$b_1, b_2, b_3, b_4, \ldots$

starts with

-1, 1, -1, 1, -1, 1, …

Exercise: What are the a_0, a_1, a_2, a_3 terms of the following sequences?

	a_0	a_1	a_2	a_3
$\{ 4^n \}$				
$\{ \lfloor \frac{n}{2} \rfloor \}$				
$\{ n! \}$				

Def. Finite sequences $a_1, a_2, a_3, \ldots a_n$ are called **strings**. A string is also denoted by $a_1\ a_2\ a_3 \ldots a_n$ (without the commas.)

Special Integer Sequences

Arithmetic sequences are those such that consecutive differences are constant. The general formula for the nth term is $\boldsymbol{a + d(n-1)}$ where a is the first term and d is the constant difference.

Example: 1, 3, 5, 7, 9, …

Find the n^{th} term (i.e. the general rule) _____

The 100^{th} term is _____

Example: 0, 50, 100, 150, 200, …

Find the n^{th} term _____

The 100^{th} term is _____

Geometric sequences: are those such that consecutive ratios (2^{nd} term divided by the 1^{st}, 3^{rd} term divided by the 2^{nd}, etc.) are constant. The general formula for the nth term is $\boldsymbol{ar^{n-1}}$ where a is the first term and r is the constant ratio.

Example: 3, 6, 12, 24, 48, …

Find the n^{th} term _____

Find the 100^{th} term _____

Example: 10, 100, 1000, 10,000, 100,000, …

Find the n^{th} term _____

The 100^{th} term is _____

Suppose that we are given a numerical sequence that is neither an arithmetic nor geometric sequence. What to do then?

Other tool to consider for discovering patterns:

- Are the terms dependant on previous terms?
- Are the terms squares? Or cubes?
- Are some terms repeated?

Example: 1, 8, 27, 64, 125, …

Find a simple formula (or general rule) for the sequence.

Example: 1, 1, 2, 3, 5, 8, 13 … (**Fibonacci sequence**)

Find a simple formula (or general rule) for the sequence.

Summations

Notation: The sum $a_1 + a_2 + a_3 + ... + a_n$ is represented by the **<u>summation notation</u>** $\sum_1^n a_i$. The variable i is called the **<u>index of summation</u>** .

$\sum_{i=1}^n a_i$ is read as " the sum for i equal 1 to n."

$\sum_{j=m}^n a_j$ is read as " the sum for j equal m to n."

$$\sum_m^n a_j$$

n is called the **<u>upper limit</u>**.

m is called the **<u>lower limit</u>**.

The Greek letter sigma is used to denote summation

Exercise: Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$

Exercise: What is the value of $\sum_{j=5}^{7} j^2$?

Exercise: What is the value of $\sum_{j=0}^{4} 2^j$?

Exercise: What is the value of $\sum_{k=1}^{5} 2^{(k-1)}$?

Double Summations

$\sum_{i=1}^{3} \sum_{j=0}^{2} ij =$

A set index summation

$\sum_{s \in \{0,1,3\}} s^2 =$

Manipulating the Notation:

- Suppose we have the sum $\sum_{j=1}^{7} j^2$ and that we need to **change the summation notation so that the lower limit is 0**. Note that this is accomplished by letting $m = j\text{-}1$. Then $j = m+1$, and

$$\sum_{j=1}^{7} j^2 = \sum_{m=0}^{6} (m+1)^2 .$$

- By associativity, $\sum_{j=1}^{6} j^2 + 7^2 = \sum_{j=1}^{7} j^2 = 1 + \sum_{j=2}^{7} j^2$
- Note: $\sum_{j=1}^{6} j^2 + 7^2 \neq \sum_{j=1}^{6} (j^2 + 7^2)$ (Beware of parenthesis)
- By the distributive property of multiplication over addition,

$$3\left(\sum_{k=1}^{n} k\right) = \sum_{k=1}^{n} 3k$$

Example: The sum of the first *n* terms of a geometric sequence is called a **geometric progression**.

Theorem $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$ if $r \neq 1$

Proof will be given in a later section.

Exercise: What is the value of $\sum_{k=1}^{n} [3 \cdot 2^{(k-1)} + 5 \cdot 2^k]$?