Functions

Def. Let A and B be sets. A **<u>function f from A to B</u>** is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f{:}A \rightarrow B$.

How is definition written in symbolic logic notation?

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Example 1:

True or False: f is a function from the set $A=\{1, 2, 3,4, 5\}$ to the set $B=\{3, 4, 5, 6\}$ if the assignment rule is $f(1) = 3, f(2) = 3, f(3) = 1, f(3) = 2, f(4) = 5.$

True or False: f is a function from the set $A=\{1, 2, 3,4, 5\}$ to the set $B=\{3, 4, 5, 6\}$ if the assignment rule is $f(1) = 3, f(2) = 3, f(3) = 4, f(4) = 5.$

True or False: f is a function from the set $A=\{1, 2, 3,4, 5\}$ to the set $B=\{3, 4, 5, 6\}$ if the assignment rule is $f(1) = 3, f(2) = 3, f(3) = 4, f(3) = 2, f(4) = 5, f(5) = 6.$

Definition: A **bit string** is a sequence of *0's* and or *1's*. Let us assume that the first bit is the rightmost bit. The **length** of a bit string is the number of bits in the string.

Example 2: Determine whether f is a function from the set of all finite length bit strings to the set of integers if

a) $f(S)$ is the position of a 1 bit in the bit string S.

b) $f(S)$ is the smallest integer i such that the i^{th} bit of S is 1.

c) $f(S)$ is the smallest integer i such that the i^{th} bit of S is 1 and $f(S) = 0$ whenever S is the empty string or the constant 0's string.

Example3: Determine whether f is a function from the set of real numbers to the set of real numbers if

a) $f(a) = \sqrt{a}$

b) $f(a) = a^2$

Definitions.

- If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f.
- If $f(a) = b$, we say that b is the **image** of a and a is a **pre-image** of b.
- The **range** of f is the set of all images of elements of A.
- Also, if f is a function from A to B, we say that ***f* maps *A* to *B*.**

Example 4: Let R be the set of real numbers. Define $f{:}R \rightarrow R$ by $f(x) = x + 1$.

In this example we see that:

1. The domain is R.
2. The codomain is R.
3. Since $f(3) = 4$, the number 4 is the image of the number 3, and 3 is the preimage of 4.
4. The range is R.

Example 5: Let Z be the set of integers, and Z^+ be the set of nonnegative integers. Define $f{:}Z \rightarrow Z$ by $f(x) = x^2$.

In this example we see that:

1. The domain is _______
2. The codomain is ______
3. Since $f(-2) = 4$, the number 4 is the _______ of the number -2, and -2 is the ________ of 4.
4. The range is Z^+. (Note that the codomain is not the same as the range).

Example 6: The **floor function** assigns to the real number x is the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.

1. What is the domain of this function?

2. What is the range of this function?

3. What is the image of 1.99?

4. What is "a" pre-image of 4?

Definition: The **ceiling function** assigns to the real number x is the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Definition. Let *f* be a function from the set *A* to the set to the set *B* and let *S* be a subset of *A*. The **image of *S*** is the subset of *B* that consists of the images of the elements of *S*. We denote the image of *S* by *f(S),* so that $f(S) = \{f(s) \mid s \in S\}$.

Example 7:

i) Let $f: R \to R$ be defined by $f(x) = 2\lfloor x \rfloor$ and $S = \{x \mid 0 < x < 3\}$. Find $f(S)$ = _______________

ii) Let $f: R \to R$ be defined by $f(x) = x^2 + 1$ and let $S = \{0, 1, 2\}$. Find $f(S)$ = _______________

iii) Let $f: R \to R$ be defined by $f(x) = 3x + 1$ and let $S = \{0, 1, 2\}$. Find $f(S)$ = _______________

Theorem Let *f* be a function from the set *A* to the set *B*. Let *S* and *T* be subsets of *A*. Then $f(S \cup T) = f(S) \cup f(T)$.

Definition. A function *f* is said to be **one-to-one**, or **injective**, if and only if $f(x) = f(y)$ implies that $x = y$ for all *x* and *y* in the domain of *f*. A function is said to be an **injection** if it is one-to-one.

Example 8:

True or False: The function *f* from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1,$ and $f(d) = 3$ is one-to-one.

True or False: The function *f* from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4, f(b) = 4, f(c) = 1,$ and $f(d) = 3$ is one-to-one.

True or False: The function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Example 9: Let S be a bit string of length n. Define $f(S)$ as the smallest integer i such that the i^{th} bit of S is 1 and $f(S) = 0$ when S is the zero string or the empty string. Is this function one-to-one?

Example 10: Prove that the real valued function $f(x) = x + 1$ is one-to-one.

Definition: A function f whose domain and codomain are subsets of the set of real numbers is called **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$ and x and y are in the domain of f.

Axiom of the Real Numbers (tricotomy) : For any two real numbers one of the following properties holds $x = y, x < y$, or $x > y$.

Theorem. Let f be a real valued function. If f is strictly increasing (or strictly decreasing), then f is one-to-one.
Proof.

Definition. A function f is said to be **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a **surjection** if it is onto.

Example: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f onto $\{1, 2, 3\}$ function?

Example 11: Determine whether the real valued function $f(x) = x + 1$ is onto.

Example 12: Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is onto.

Definition. The function f is called a **one-to-one correspondence**, or a **bijection** if it is both one-to-one and onto.

Definition. Let f be a bijection from the set A to the set B. The **inverse function of *f*** is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{1}. Hence, $f^{1}(b) = a$ when $f(a) = b$.

Terminology: A one-to-one correspondence is called **invertible**.

Example 13: Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$. Is f an invertible function? If so describe the inverse function.

Example 14: Let $f : Z \to Z$ defined by $f(x) = x + 1$. Is f invertible? If so describe the inverse function.

Example 15: Let $f : Z \to Z$ defined by $f(x) = x^2$. Is f invertible? If so describe the inverse function.

Definition. Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition of the functions f and g**, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.

Example 16: Let $f : Z \to Z$ and $g : Z \to Z$ defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Definition: Let A be a set. The **identity function on A** is the function $i_A : A \to A$ where $i_A(x) = x$.

Fact: Let $f : A \to B$ be an invertible function. If $f(a) = b$, then
$(f^{-1} \circ f)(a) = a$ and
$(f \circ f^{-1})(b) = b$.

Graphs of Functions

Definition. Let $f: A \to B$. The **graph of the function *f*** is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$ (A graph is a subset of the Cartesian product $A \times B$)

Example: Let $f: Z \to Z$ be defined by $f(n) = 2n + 1$. Display the graph of *f*.

Example: Let $f: R \to Z$ be defined by $f(x) = \lfloor x \rfloor$. Display the graph of *f*.

Exercise :
Let *S* be a subset of a universal set *U*. The **characteristic function** f_S of *S* is the function from *U* to the set *{0, 1}* (i.e. $f_S : U \to \{0,1\}$) such that $f_S(x) = 1$ if *x* belongs to *S* and $f_S(x) = 0$ if *x* does not belong to *S*. Let *A* and *B* be sets. Show that for all *x*

a) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$
Proof: You give the reasons.

$f_{A \cap B}(x) = 1$	$\Leftrightarrow x \in A \cap B$	
	$\Leftrightarrow x \in A \wedge x \in B$	
	$\Leftrightarrow f_A(x) = 1 \wedge f_B(x) = 1$	
	$\Leftrightarrow f_A(x) \cdot f_B(x) = 1$	