Predicates and Quantifiers

Recall that the expression $x > 3$ is not a proposition. Why?

P(x) **Notation**:

- We will use the <u>propositional function</u> notation $P(x)$ to denote the expression " *x has property P*".
- As soon as x is assume to be some value then $P(x)$ has a truth value.

The goal of the section is to quantify such statements so that the result is a proposition. The process is very much in the spirit of the next example.

Example: Let $P(x)$ denote the statement "$x > 3$".
What are the truth values:
i. $P(4)$ True or False ?
ii. $P(2)$ True or False ?
iii. $P(y)$ True or False ?

Example: Let $Q(x,y)$ denote the statement "$x = y + 3$".
What are the truth values:
i. $Q(7,4)$ True or False ?
ii. $Q(2,2)$ True or False ?

Quantifiers

Def. The <u>**universe of discourse**</u> for a math statement is the domain of that statement.

Def. The <u>**universal quantification of *P(x)***</u> is the proposition
" *P(x) is true for all values of x in the universe of discourse*."

- This proposition is denoted by $\forall x\ P(x)$.
- The proposition $\forall x\, P(x)$ is read as "for all x $P(x)$" or "for every x $P(x)$".
- The symbol $\forall$ is called the <u>**universal quantifier**</u>.

Def. The <u>**existential quantification of *P(x)***</u> is the proposition
" *There exists an element x in the universe of discourse for which P(x) is true*."

- This proposition is denoted by $\exists x\, P(x)$.
- The symbol $\exists$ is called the <u>**existential quantifier**</u>.
- The proposition $\exists x\, P(x)$ is read as "for some x $P(x)$" or "there exists an x such that $P(x)$".

The Truth Values of $\forall x\, P(x)$ and $\exists x\, P(x)$

Example 1: Suppose that the universe of discourse for x is the set of nonnegative integers $\{0, 1, 2, 3, 4, 5, \ldots\}$

What is the truth value of $\forall x\ (x > 0)$? ________

What is the truth value of $\exists x\ (x > 0)$? ________

Example 2: Suppose that the universe of discourse for $P(x)$ is the small set $\{ x_1, x_2, x_3, x_4 \}$.

Since

$$\forall x\, P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge P(x_4)$$

when is $\forall x\, P(x)$ true? ____________________________________

when is $\forall x\, P(x)$ false? ____________________________________

Since

$$\exists x\, P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee P(x_4)$$

when is $\exists x\, P(x)$ true? ____________________________________

when is $\exists x\, P(x)$ false? ____________________________________

Note: The above observations also hold for infinite universe of discourse sets.

Examples 3:

Where $P(x)$ denotes:	Universe of Discourse for $P(x)$	Truth value of $\exists x\, P(x)$	Truth value of $\forall x\, P(x)$
$x = x + 3$	Real numbers		
$x = x + 0$	Real numbers		
$0/x = 0$	Integers		
$0 - x = x$	Positive Integers		
$0 - x = x$	Natural numbers		

Example 4:

(i) What is the truth value of $[\forall x P(x) \rightarrow \exists x P(x)]$? ______________

(ii) What is the truth value of $[\exists x P(x) \rightarrow \forall x P(x)]$? ______________

Example 5: Let *P(x)* denote the statement "$x^2 = 4$". Suppose that the universe of discourse is the set of real numbers.

- Which of the following is the English equivalent to $\exists x\, P(x)$

(a) There is a real number *x* such that $x^2 = 4$.
(b) For every real number *x* it is the case that $x^2 = 4$.

- What is the truth value of $\exists x\, P(x)$?

Example 6: Let *P(x,y)* denote the statement " $x^2 = y$ ". Suppose that the universe of discourse is the set of real numbers.

- Which of the following is the English equivalent to $\forall x\, P(x)$?

(c) There is a real number *x* such that $x^2 = 4$.
(d) For every real number *x* it is the case that $x^2 = 4$.

- What is the truth value of $\forall x\, P(x)$?

Definition. A real number *x* is a **rational number** if can be expressed as the ratio of two integers *a* and *b* with b not zero, that is if $x = \frac{a}{b}$, with $b \neq 0$.

Example 7: Let *Q(x)* denote the statement " *x* is a rational number ", and *I(y)* denote the statement " *y* is an integer. Assume the universe of discourse is the set of real numbers.

i. $\exists y\, (Q(y) \wedge I(y))$

- Which of the following is the English translation?

(a) There is a rational number *y* which is a rational number and also an integer.
(b) There is a real number *y* which is a rational number and also an integer.
(c) Every real number *y* is a rational and also an integer.

- What is the truth value? ____________

ii. $\forall y\, (Q(y) \wedge I(y))$

- Which of the following is the English translation?

(a) There is a rational number *y* which is a rational number and also an integer.
(b) There is a real number *y* which is a rational number and also an integer.
(c) Every real number *y* is a rational or it is an integer.
(d) Every real number *y* is a rational and also an integer.

- What is the truth value? ____________

iii. $\forall y\,(I(y) \rightarrow Q(y))$

- Which of the following is the English translation?

(a) Every real number y is an integer and also a rational number.
(b) There is a real number y such that if y is an integer then y is a rational number.
(c) For any real number y, if y is an integer then y is a rational number.

- What is the truth value? ___________

iv. $\forall y\,(\neg I(y) \rightarrow Q(y))$

- What is the English translation? ____________________________
- What is the truth value? ___________

v. $\forall y\,(Q(y) \rightarrow I(y))$

- What is the English translation? ____________________________
- What is the truth value? ___________

Example 8: Assume the universe of discourse is the set of all students at UHD. Let *C(x)* be " *x* has a computer" and let *F(x,y)* be " *x* and *y* are friends."

i. Translate $C(Judy)$ into English. ________________________

ii. Which is the English translation of $\exists y(C(y) \wedge F(Judy, y))$.
(a) Judy has a friend who has a computer.
(b) Judy has a computer and a friend.
(c) There is a student who has a friend and a computer.

iii. Translate $C(Judy) \wedge \exists y(C(y) \wedge F(Judy, y))$ into English.

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iv. Translate $\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y))$ into English.

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Negations of Quantified Expressions

Example 9: Consider the proposition " *Every student in the class has taken a course in calculus*."

- When is this proposition true? _________________

- Write the proposition in symbolic logic notation.
  ____
  ____

- Which of the following is the negation of "*Every student in the class has taken a course in calculus*."
  (a) No student in the class has taken calculus.
  (b) Some students in the class have not taken calculus
  (c) Some student in the class has not taken calculus

- Write the negation of the proposition in symbolic logic notation.
  ____
  ____

- When is the negation of the proposition "*Every student in the class has taken a course in calculus*" true? _______________________

Fact:

$$\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$$
$$\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

Example 10:

(iii) What is the truth value of $[\forall x P(x)] \vee [\neg\forall x P(x)]$? _____________

(iv) What is the truth value of $[\forall x P(x)] \vee [\exists x \neg P(x)]$? _____________

(v) What is the truth value of $[\forall x P(x)] \wedge [\exists x \neg P(x)]$? _____________

Example 11: Show that $\neg\exists x(P(x) \vee Q(x)) \equiv \forall x(\neg P(x) \wedge \neg Q(x))$.