Propositional Equivalences

Def. A compound proposition that is always true, no matter what the truth values of the (simple) propositions that occur in it, is called **tautology**. A compound proposition that is always false, no matter what, is called a **contradiction**. A proposition that is neither a tautology nor a contradiction is called a **contingency**.

Examples: Let p be a proposition. Indicate whether the propositions are:
(A) tautologies (B) contradictions or (C) contingencies.

Proposition	type
$p \vee \neg p$	
$p \wedge \neg p$	
$x+7=18$ for every real number x	

Def. The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent. Some text books use the notation $p \Leftrightarrow q$ to denote that p and q are logically equivalent.

Objective of the section:

You must learn to determine if two propositions are logically equivalent by the

- truth table method **and**
- by the logical proof method (using the tables of logical equivalences.)

Exercise 1: Use truth tables to show that $\neg\neg p \equiv p$ (the double negation law) is valid.

Exercise 2: Use truth tables to show that $p \wedge \mathrm{T} \equiv p$ (an identity law) is valid.

Note: Any equivalence termed a "law" will be proven by truth table, but all others by proof (as we shall see next).

Equivalence	*Name*
$p \wedge \mathrm{T} \equiv p$ $p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}$ $p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg\neg p \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee \neg p \equiv \mathrm{T}$ $p \wedge \neg p \equiv \mathrm{F}$ $(p \to q) \equiv \neg p \vee q$ $(p \to q) \equiv \neg q \to \neg p$	Negation laws Other useful logical equivalence

Exercise 3: State the name of the law used in the identity

i. $\neg(\neg p \wedge q) \vee \mathrm{T} \equiv \mathrm{T}$ ______________________

ii. $\mathrm{T} \vee \neg(\neg p \wedge q) \equiv \neg(\neg p \wedge q) \vee \mathrm{T}$ ______________________

iii. $\neg(\neg p \wedge q) \wedge \mathrm{T} \equiv \neg(\neg p \wedge q)$ ______________________

iv. $\neg(\neg p \wedge q) \equiv \neg\neg p \vee \neg q$ ______________________

Exercise 4: ***Without*** truth tables to show that

$$\neg(\neg p \wedge q) \equiv p \vee \neg q$$

Exercise 5: ***Without*** truth tables to show that

$$[\neg(p \wedge q) \vee (p \wedge q)] \equiv \mathbf{T}$$

Exercise 6: ***Without*** truth tables to show that

$$\neg p \wedge (p \vee q) \equiv \neg p \wedge q$$

Exercise 7: ***Without*** truth tables to show that

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg(p \vee q)$$

Exercise 8: ***Without*** truth tables to show that $(\neg p \rightarrow q) \equiv p \vee q$.

Exercise 9: ***Without*** truth tables to show that $\neg (p \rightarrow q) \equiv p \wedge \neg q$.

Exercise 10: ***Without*** truth tables to show that $\neg (\neg p \vee (p \vee q)) \rightarrow q$ is a tautology.

Exercise 11: ***Without*** truth tables to show that an implication and it's contrapositive are logically equivalent.

Applications

In addition to providing a foundation for theorem proving, which we will cover in this class, this algebraic look at logic can be studied further for the purpose of discussion computer program correctness.