Logic

Def. A **Proposition** is a statement that is either true or false.

Examples: Which of the following are propositions?

Statement	Proposition (yes or no)	If yes, then determine if it is true or false.
UHD has a brick exterior		
$1 + 3 = 0$		
$2 + 3$		
$x + 7 = 18$		
$x+7=18$ for $x = 9$		
$x+7=18$ for every real number x		

Notation: We use lower case letters to denote propositions.

Def. **Compound propositions** are new propositions formed from existing propositions using *logical operators*.

Note: We will discuss the following *logical operators* (also called connectives)

- Negation operator, $\neg$ (Other books use ~)
- Conjunction operator, $\wedge$
- Disjunction operator, $\vee$
- Exclusive or, $\oplus$
- Implication, $\rightarrow$
- Biconditional, $\leftrightarrow$

Def. Let p be a proposition. The statement " it is not the case that p" is a proposition formed from p and the negation operator, called **the negation of *p***, which we denote by $\neg p$. This proposition is read as "not p". The truth-value of $\neg p$ is true when p is false and false when p is true.

Truth Table

p	$\neg p$
T	F
F	T

Examples:

1. Let p be the statement "$7 + 1 = 6$".
    a. Write the statement $\neg p$ as an English statement.

    b. What is the truth-value of $\neg p$?

2. Let p be the statement "*Bill Clinton was the president of the U.S in 2006*".
    a. Write the statement $\neg p$ as an English statement.

    b. What is the truth-value of $\neg p$?

Def. Let p and q be propositions. The statement "p and q" is a proposition formed from p, q and the conjunction operator, called **<u>the conjunction of *p* and *q*</u>**, which we denote by $p \wedge q$. The truth-value of $p \wedge q$ is true when both p and q are true, otherwise $p \wedge q$ is false.

Truth Table

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

Def. Let p and q be propositions. The statement "p or q" is a proposition formed from p, q and the disjunction operator, called **<u>the disjunction of *p* and *q*</u>**, which we denote by $p \vee q$. The truth-value of $p \vee q$ is false when both p and q are false, otherwise $p \vee q$ is true.

Truth Table

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

Examples: Let p be the statement "*Houston is the capital of Texas*" and let q be the statement "*Houston is a city in Texas*". Determine the following:

a. The truth-value for $p \wedge q$.
b. The truth-value for $p \vee q$.
c. The truth value for $\neg p \wedge q$
d. The truth value for $\neg (p \vee q)$
e. The truth value for $\neg p \vee q$

Def. Let p and q be propositions. The statement "p or q exclusively" is a proposition formed from p, q and the exclusive or operator, called the **exclusive or of p and q**, which we denote by $p \oplus q$. The truth-value of $p \oplus q$ is true when exactly one of p or q is true, otherwise $p \oplus q$ is false.

Truth Table

p	q	$p \oplus q$
T	T	
T	F	
F	T	
F	F	

Comparison: A familiar example of the use of an "exclusive or" is a restaurant menu. "Price of the entree includes soup or salad". Note that one or the other is included in the price *but not both*. A familiar example of the use of inclusive or is the prerequisite for this course. "Students who have taken calculus or computer science, can take this course." Note that if a student has taken both courses, then the student may still take this course.

Def. An **implication** (or **conditional statement**) is statement of the form "if p then q", denoted $p \to q$. Such propositions are also read as "p implies q". The "if part" is called the **hypothesis** (or **premise**), and the "then part" is called the **conclusion** (or **consequence**). The implication if p then q is false only when the hypothesis is true and the conclusion is false.

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Terminology: An implication $p \to q$ may be worded in several different ways:

- If p, then q
- If p, q
- q if p
- p only if q
- p is sufficient for q
- q is necessary for p

Examples:

a. True or False : If *2* is an even integer, then today is Friday.

b. Write the statement in the form "if p, then q" and determine the truth-value. $1 + 1 = 2$ *if* $3 + 3 = 7$.

c. Write the statement in the form "if p, then q" and determine the truth-value. $1 + 1 = 2$ *only if* $3 + 3 = 7$.

Def. Given the implication $p \to q$:

$q \to p$ is the **converse** of the implication $p \to q$.
$\neg p \to \neg q$ is the **inverse** of implication $p \to q$.
$\neg q \to \neg p$ is the **contrapositive** of implication $p \to q$.

Example: Write the converse, the inverse, and the contrapositive of the given implication:

If *cows eat grass*, then $2 + 3 = 4$.	Proposition	True or False?
	Converse	True or False?
	Inverse	True or False?
	Contrapositive	True or False?

Def. The **biconditional proposition** denoted by $p \leftrightarrow q$ is the conjunction of $p \to q$ and $q \to p$. The proposition $p \leftrightarrow q$ is read as p if and only if q. The biconditional proposition is true when both p and q have the same truth-values and is false otherwise.

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Examples: Determine the truth-values for the following:

a. The number *2* is an odd integer if and only if today is Friday. ____
b. If $3 + 3 = 7$, $1 + 1 = 2$ and conversely if $1 + 1 = 2$, $3 + 3 = 7$. ____

Exercise. Let *p* and *q* be the propositions

p: It is below freezing.

q: It is snowing

Write the following propositions using the symbols *p* and *q* and any appropriate logical connectives.

Use the following (a) –(d) to fill in the blank for i) and ii)

(a) $p \wedge q$ (b) $p \vee q$ (c) $p \to q$ (d) $q \to p$ (e) none of these

i) It is below freezing and snowing. ________

ii) It is below freezing if it is snowing ________

Use the following (a) –(d) to fill in the blank for iii) - vi)

(a) $p \oplus q$ (b) $p \vee q$ (c) $p \to q$ (d) none of these

iii) It is not below freezing and it is not snowing. ________

iv) It is either below freezing or snowing (or both). ________

v) It is either below freezing or snowing (but not both). ________

vi) It is below freezing if and only if it is snowing. ________

Exercise Construct a truth table for each of the following compound propositions. Note: your truth-table will require 2^n rows, where *n* is the number of simple propositions in the compound proposition.

a) $(p \wedge q) \vee \neg q$

b) $(p \vee q) \wedge \neg r$